

The Home of the Constants

— 4π and $32\pi^2$ as Index-Theorem Normalizations —

Placing the Yamagishi Patch in the Atiyah–Singer Family, Without Merging Its Branches

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ORCID: 0009-0006-8587-3450 | **Paper 70** | **Zenodo:** 10.5281/zenodo.21650347

Category: Mathematical Physics / Topology / Structural Interpretation

Keywords: Atiyah–Singer index theorem, Gauss–Bonnet–Chern, Euler class, Chern character, Poincaré–Hopf, characteristic classes, distinct roots

— *From thinking to syncing.* —

Positioning of this paper

This paper offers a **structural interpretation**: it identifies the mathematical *home* of the recurring constants of the Yamagishi system (4π , $16\pi^2$, $32\pi^2$, and the $(4\pi)^n n!$ ladder) as normalization constants of characteristic classes in the Atiyah–Singer index theorem. It claims no new mathematics (the index theorem is assumed and cited) and no new physics. Crucially, following the discipline of Papers 48 and 58, it shows the index theorem is a *home, not a melting pot*: it unifies the *statement* "analytic index = topological index" across its branches, while the numerical constants of different branches (Euler, Chern-character, Dirac) remain **distinct roots**. No claim contradicting established theory is made.

ABSTRACT

Paper 67 anchored the cornerstone as $4\pi = 2\pi\chi(S^2)$ (the 2D Gauss–Bonnet case); Paper 48 unpacked the higher normalization $(4\pi)^n n!$ of Chern–Gauss–Bonnet in $2n$ dimensions. This paper names their common home. Gauss–Bonnet–Chern is one branch of the **Atiyah–Singer index theorem**: for an elliptic operator, the analytic index equals a topological integral of a characteristic class. Three points. (i) *Placement*: $4\pi = 2\pi\chi(S^2)$ and $(4\pi)^n n!$ are the *Euler-class* branch normalizations; Poincaré–Hopf (Hopf 1926) is the prototype of "index." (ii) *Two rooms, one house*: the instanton $16\pi^2$ of Paper 45 (the second Chern class, $k = \frac{1}{16\pi^2} \int \text{tr } F \wedge F$) also lives in the theorem, but in the *Chern-character* branch of the gauge-coupled Dirac operator—a different branch. The index theorem therefore both unifies (a common home) and *certifies the distinctness* Paper 48 found between the Euler $32\pi^2$ and the instanton $16\pi^2$. (iii) *Consequence*: the constants are named members of a classical family, and their topological status (Paper 67) extends to all even dimensions via Paper 48. This is a placement, not a derivation; the spin/Dirac ($\hat{A}$ -genus) branch—linking to the qubit of Paper 68—is flagged as a direction, not a claim.

1. Introduction

Two earlier results placed the recurring π -power constants on topological ground. Paper 67: $4\pi = 2\pi\chi(S^2)$ (the cornerstone 4π is 2π times the Euler characteristic, fixed by Poincaré–Hopf). Paper 48: the Chern–Gauss–Bonnet normalization in $2n$ dimensions is $(4\pi)^n n!$ ($4\pi, 32\pi^2, 384\pi^3$), and this Euler $32\pi^2$ is a distinct root from the instanton $16\pi^2$ of Paper 45.

This paper supplies the keystone: the common home is the **Atiyah–Singer index theorem**. Naming the home certifies that the patch constants are topological normalizations, exhibits the Euler and instanton constants as living in different branches (which is exactly why they are distinct roots), and identifies the spin/Dirac branch as the next door (to Paper 68).

What this paper does not do. It does not prove the index theorem (assumed, cited) and adds no physics. It places known constants in a known family and does *not* merge the branches.

2. The Index-Theorem Family (Established)

The Atiyah–Singer index theorem (1963) states that for an elliptic operator D on a closed manifold M ,

$$\underbrace{\text{index}(D)}_{\text{analytic: } \dim \ker D - \dim \text{coker } D} = \underbrace{\int_M [\text{a characteristic class}]}_{\text{topological}}$$

Different D give the classical theorems, each with its own characteristic class and normalization:

Operator / theorem	Characteristic class	Computes	Branch
de Rham / Euler (Gauss–Bonnet–Chern)	Euler class $e(TM)$	$\chi(M)$	Euler
Signature (Hirzebruch)	L -genus	signature $\sigma(M)$	signature
Dolbeault (Riemann–Roch)	Todd class	holomorphic Euler char.	Todd
Dirac (spin)	$\hat{A}$ -genus	spinor index	Dirac
gauge-coupled Dirac	$\hat{A}(M) \text{ ch}(E)$	incl. instanton no.	Chern-character

The word "index" descends from the *index of a vector field*: Poincaré–Hopf (Hopf 1926) states that the sum of vector-field indices on M equals $\chi(M)$, the index of the de Rham (Euler) operator. Hopf's theorem is thus the prototype of "index."

3. Where the Patch Constants Sit

Proposition 3.1 (The patch constants are Euler-branch normalizations)

The constants of Papers 67 and 48 are the Euler-class branch normalizations:

- In 2D, $\int_{S^2} K dA = 2\pi\chi(S^2) = 4\pi$ (Paper 67): Gauss–Bonnet, $\int_M e(TM) = \chi(M)$, normalization 2π .
- In $2n$ dimensions, the Chern–Gauss–Bonnet normalization is $(4\pi)^n n!$ (Paper 48): 4π , $32\pi^2$, $384\pi^3$ for $n = 1, 2, 3$.

Thus 4π and $32\pi^2$ are the $n = 1, 2$ Euler-class normalizations, and Poincaré–Hopf is their vector-field-index prototype.

Proof. Both are quoted from the cited papers (Paper 67 for the $n = 1$ Gauss–Bonnet identity, Paper 48 for the $(4\pi)^n n!$ normalization). Placing them in the index theorem is the identification of Gauss–Bonnet–Chern as the Euler branch of Atiyah–Singer (standard). ■

Proposition 3.2 (The instanton constant is a different branch)

The $16\pi^2$ of Paper 45—the second Chern class, entering $k = \frac{1}{16\pi^2} \int_M \text{tr}(F \wedge F)$ —is the *Chern-character* branch normalization, governing the gauge-coupled Dirac operator (its index integral contains $\text{ch}(E)$, hence c_2 and the instanton number). This is *not* the Euler branch of Proposition 3.1.

The two constants $32\pi^2$ (Euler) and $16\pi^2$ (Chern-character) sit in the *same theorem* but in *different branches*, built from different characteristic classes ($e(TM)$ vs. $c_2(E)$).

4. A Home, Not a Melting Pot

Observation 4.1 (The index theorem unifies the statement, not the numbers)

Every row of Section 2 is an instance of one statement (analytic index = topological integral). In this sense the index theorem *unifies* the patch constants. But the *characteristic classes differ by branch*: the Euler class e , the L -genus, the Todd class, the $\hat{A}$ -genus, and the Chern character are distinct cohomology classes with distinct normalization constants. Therefore the numbers are **not** unified: the Euler $(4\pi)^n n!$, the instanton $16\pi^2$, and the Dirac $\hat{A}$ constants are distinct roots.

This is precisely the discipline of Papers 48 and 58, seen from above. Paper 48 found the Euler $32\pi^2$ and the instanton $16\pi^2$ to be distinct roots and warned against identifying them. The index theorem *explains why*: they are normalizations of different characteristic classes (e vs. c_2) that happen to appear in the same grand theorem. **A common home is not a common origin. The index theorem is the family house; the constants are siblings, not the same person.**

5. Consequences

Corollary 5.1 (The patch constants are certified topological, in all even dimensions)

By Proposition 3.1 the Euler-branch constants are topological normalizations in every even dimension; Paper 67's 2D statement (4π is topological) is not isolated but the base of a tower. The constants are not arbitrary; they are the values a topological invariant must take.

Observation 5.2 (The spin/Dirac door)

The one branch not yet touched by the system is the Dirac ($\hat{A}$ -genus) branch. It is the natural sequel, because the qubit of Paper 68 already lives in the spin world ($SU(2)$, the Bloch sphere, the Berry phase). Whether the constants acquire an $\hat{A}$ -genus reading is left as a *direction*, not a claim—pursued, if at all, through the spin door as with Paper 68, never by forcing the Euler and Dirac constants together.

6. Scope of the Claims

What is established

1. 4π and $(4\pi)^n n!$ (Papers 67, 48) are the Euler-branch normalizations of Atiyah–Singer; Poincaré–Hopf is the prototype of "index" (Prop. 3.1).
2. The instanton $16\pi^2$ (Paper 45) is the Chern-character-branch normalization—a different branch (Prop. 3.2).
3. The index theorem unifies the statement but not the numbers; the branch constants are distinct roots, explaining Paper 48's finding from above (Obs. 4.1).

What is not established / Open problem

- Any new mathematics (Atiyah–Singer and its special cases are assumed and cited).
- Any new physics, or that the index theorem "derives" the system. This is a *placement* of known constants in a known family.
- Any merging of branches. The Euler, Chern-character, and Dirac constants remain distinct roots; the $\hat{A}$ -genus reading is a direction only.
- *Open*: does any Yamagishi construction correspond to a *single* elliptic operator whose index is being computed—so a patch constant is literally $\text{index}(D)$, not only a normalization? If so, the placement becomes a genuine index computation.

7. Conclusion

The recurring constants have a classical home: they are normalization constants of characteristic classes in the Atiyah–Singer index theorem. The Euler-branch $4\pi = 2\pi\chi(S^2)$ and $(4\pi)^n n!$ (Papers 67, 48) and the Chern-character instanton $16\pi^2$ (Paper 45) live in the same theorem but in different branches, built from different characteristic classes—so the index theorem unifies the statement "analytic index = topological index" while leaving the numbers as distinct roots, exactly as the discipline of Papers 48 and 58 requires. The patch constants are thereby certified topological in all even dimensions and shown to be named members of a venerable family, whose spin/Dirac room—reached, as ever, through the spin door of Paper 68—remains open for a later visit.

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