

Born's Exponent and Gleason's Theorem

— Placing the Geometric Reading Beneath the Stronger Derivation —
and the Qubit, Where Gleason Is Silent

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— *From thinking to syncing.* —

Positioning of this paper

This paper offers a **structural interpretation** and, above all, an act of *intellectual humility*. It does not derive the Born rule better than the standard result; on the contrary, it places the geometric reading of the exponent~2 from Papers 53 and 68 *beneath* Gleason's theorem, which derives Born's rule from strictly weaker assumptions (non-contextuality and additivity). It records one genuinely interesting point in the system's favour: the qubit—the Hopf case of Paper 68—is exactly the two-dimensional case where projective Gleason *fails*, so the geometric picture there addresses precisely Gleason's blind spot (restored, rigorously, only by the POVM version). It does not solve the measurement problem and claims no new physics. No claim contradicting established theory is made.

ABSTRACT

Papers 53 and 68 read the exponent~2 of Born's rule as a sesquilinear "2": $p = |\psi|^2 = \langle \psi | P | \psi \rangle$, the same quadratic form as the Hopf (Bloch) map. This paper asks the honest question—is "why 2" *derived* or merely *assumed*?—and places the geometric reading beneath the rigorous royal-road result. **Gleason's theorem** (1957) shows that on a Hilbert space of dimension ≥ 3 , any non-contextual, countably additive measure on the projection lattice has the form $\mu(P) = \text{Tr}(\rho P)$; for a pure state this is Born's rule, exponent~2 included, *derived* from assumptions strictly weaker than assuming $|\psi|^2$. We record a clear hierarchy: Gleason (and, for all dimensions including the qubit, the POVM extension of Busch, 2003) is the derivation; the readings of Papers 53 and 68 are a *geometric picture*—subordinate as a derivation, complementary as a picture. We then note the one place the picture is not subsumed: projective Gleason requires **dim** ≥ 3 and *fails for the qubit (dim 2)*, exactly the Hopf case of Paper 68. Finally we close the "census of twos": Gleason explains *why* the sesquilinear form is forced (non-contextuality), grounding the shared root Born=Hopf. Neither Gleason nor this paper addresses the measurement problem, and the distinct-roots discipline is preserved.

1. Introduction

Paper 53 argued for the exponent ~ 2 from two structural routes (unitary conservation of $\int |\psi|^2$; the r^{-2} flux of an outgoing spherical wave), while explicitly noting the result is weaker than Gleason's theorem, which needs no assumption that the density is a power of $|\psi|$. Paper 68 identified this exponent with the quadratic Hopf (Bloch) map: Born's " 2 " and the Hopf map's " 2 " are the same sesquilinear root $\langle \psi | A | \psi \rangle$.

This paper takes the honest next step. If Papers 53 and 68 give a *reading* of the exponent, what *derives* it? The answer is a classical result—Gleason's theorem—and the disciplined thing is to place the geometry beneath it, not beside it. In doing so we find one point genuinely in the picture's favour: the qubit, the very object of Paper 68, is the dimension where projective Gleason does not apply.

What this paper does not do. It does not out-derive Gleason (it defers to it), does not address the measurement problem, and adds no physics. It arranges known results in the correct order of strength.

2. Gleason's Theorem (Established)

Theorem 2.1 (Gleason, 1957)

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$. Let μ assign to each projection P a number $\mu(P) \in [0, 1]$ with $\mu(\mathbb{1}) = 1$, countably additive over mutually orthogonal projections (*non-contextuality*: $\mu(P)$ does not depend on which orthogonal resolution P belongs to). Then there is a unique density operator ρ with

$$\mu(P) = \text{Tr}(\rho P) \quad \text{for all } P.$$

For a pure state $\rho = |\psi\rangle\langle\psi|$ and rank-one $P = |\varphi\rangle\langle\varphi|$,

$$\mu(P) = \langle \psi | P | \psi \rangle = |\langle \varphi | \psi \rangle|^2$$

which is Born's rule, *including the exponent 2*. The exponent is thus not an axiom but a consequence of the trace form, which itself follows from non-contextuality and additivity. Gleason's non-contextuality is the same structural input whose

deterministic version is forbidden by the Kochen–Specker theorem (1967); Gleason uses it for measures, where it is consistent, and yields Born uniquely.

3. The Honest Hierarchy

Proposition 3.1 (Order of strength)

Regarding the exponent~2 of Born's rule:

1. **Gleason (derivation).** On $\dim \geq 3$, the exponent~2 is derived from non-contextuality and additivity (Theorem 2.1). No assumption "probability = $|\psi|^{\text{power}}$ " is made.
2. **Papers 53, 68 (picture).** Paper 53 assumes the density is a power of $|\psi|$ and singles out the power~2; Paper 68 assumes the sesquilinear form $\langle \psi | A | \psi \rangle$ and reads its "2" as the Hopf map. Both are *weaker as derivations*—they assume part of what Gleason concludes—but supply a *geometric picture* Gleason does not: on the qubit, "2" is the quadratic Hopf map, and Born's probability is the pole projection $p_0 = (1 + n_z)/2$ of the Bloch vector.

The correct arrangement is hierarchical, not parallel: Gleason is the derivation; Papers 53 and 68 are the picture beneath it.

This is the disciplined reading. The system's contribution to "why 2" is not a competing derivation but a geometry; **deferring to Gleason is not a defeat but the placing of one's own work in the correct order of logical strength.**

4. The Qubit: Where Gleason Is Silent

Observation 4.1 (Projective Gleason fails in dimension two)

Theorem 2.1 requires $\dim \mathcal{H} \geq 3$. It *fails* for the qubit ($\dim \mathcal{H} = 2$): on $\mathbb{C}^2$ the only orthogonal resolutions of the identity are pairs $\{P, 1 - P\}$, so additivity links each projection only to its unique complement, leaving room for non-contextual measures *not* of the trace form. Projective Gleason thus does not derive Born's rule for a single qubit.

This is exactly the object of Paper 68: the qubit whose state space is the Hopf fibration $S^3 \rightarrow S^2$. **The Gleason-exceptional case and the Hopf case coincide.** Two consequences—one honest concession, one honest credit.

- *Concession.* The rigorous derivation of Born *for the qubit* is not projective Gleason but its POVM extension: Busch (2003) and Caves–Fuchs–Manne–Renes (2004) show a non-contextual measure on the positive effects is $\text{Tr}(\rho E)$ in every dimension, qubit included. Born on the qubit is derived—by POVM-Gleason, not the projective theorem.
- *Credit.* Because projective Gleason is silent at **dim 2**, the geometric reading of Paper 68 is not there subsumed by it. On the one object where the classical projective theorem says nothing, the system offers a picture: the exponent **2** is the quadratic Hopf map, measurable through the Berry phase (Paper 68). **The picture earns its keep precisely in Gleason's blind spot.**

5. Closing the Census of Twos

Paper 68 assembled a "census of twos" and found a single shared root: Born's **2** and the Hopf map's **2** are both the sesquilinear form $\langle \psi | A | \psi \rangle$. That census identified the coincidence but did not explain *why* the sesquilinear form should be the one physics uses.

Corollary 5.1 (Gleason grounds the shared root)

Gleason supplies the missing "why": on **dim ≥ 3** (and, via the POVM extension, in every dimension), non-contextuality and additivity *force* the measure to be $\text{Tr}(\rho P)$ —the sesquilinear form. The shared root of Paper 68 (Born = Hopf = sesquilinear) is therefore not an accident of notation but a consequence of non-contextuality. The other twos of Paper 68 (spatial **n = 2**, topological **$\chi = 2$** , trace-reversal **2**, spinor **2**) remain distinct roots; Gleason grounds only the sesquilinear one.

6. Scope of the Claims

What is established

1. Born's exponent~2 is derived by Gleason (**dim** ≥ 3) and, for all dimensions including the qubit, by the POVM extension (Theorem 2.1; Observation 4.1).
2. The readings of Papers 53 and 68 are subordinate to Gleason as derivations, complementary as a geometric picture (Proposition 3.1).
3. Projective Gleason is silent for the qubit—the Hopf case of Paper 68 (Observation 4.1); Gleason grounds the sesquilinear shared root of the census (Corollary 5.1).

What is not established / Open problem

- Any improvement on Gleason (this paper defers to it; the geometry is a picture, not a rival derivation).
- Any solution to the measurement problem (Gleason and this paper derive/read the probability *rule*; neither explains the occurrence of a definite outcome).
- Any merging of the census twos beyond the single sesquilinear root grounded by Gleason.
- *Open*: is there a non-contextuality-like principle formulated *intrinsically on the Hopf sphere* that would derive Born for the single qubit geometrically—recovering, from $S^3 \rightarrow S^2$, what POVM-Gleason supplies algebraically? If so, the picture of Paper 68 would become, on its own turf, a derivation.

7. Conclusion

Born's exponent~2, read in Papers 53 and 68 as the sesquilinear/Hopf "2," is *derived*—not by those readings but by Gleason's theorem, from non-contextuality and additivity, on Hilbert spaces of dimension at least three, and in every dimension by its POVM extension. The honest arrangement is hierarchical: Gleason is the derivation, the geometry is the picture beneath it. The one place the picture is not subsumed is the qubit, where projective Gleason falls silent and the Hopf reading of Paper 68 addresses precisely that gap. Gleason also grounds the census of Paper 68 by forcing the sesquilinear form. Neither Gleason nor this paper touches the measurement problem, and the distinct roots stay distinct. **To defer to the stronger theorem, and to keep only the geometry that survives beneath it, is the discipline of the system exercised on its own quantum cornerstone.**

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