There Are Two Ways to Stiffen Paper, One Using More Paper and One Not, and in Stiffness the Two Are Exactly Identical

— raising the grammage from **80** to **100** g/m² and lowering the density by **20** % at the same grammage both give **0.125000** mm and the same stiffness factor of **1.953125** — the separator is whether the quantity wanted is mass per unit area or bending stiffness — [Paper 732]

Yuuki YAMAGISHI (山岸 勇輝)
Yamagishi Complexity Torus Lab, Japan
September 2026

— *From thinking to syncing.* —

Abstract

The substance of paper is stated in grammage. That is mass per unit area, not stiffness.Two ways of stiffening it are shown to be indistinguishable in stiffness.No new theorem or law is claimed. First, bending stiffness goes as the cube of thickness——80 g/m² at 800 kg/m³ gives 0.100000 mm and 2.500000 × 10^{-4} N m (Section 2). **Second, 25 % more grammage nearly doubles the stiffness——a factor of 1.953125, matching the cube of the thickness ratio 1.25** (Section 2). **Third, and this is the core. Lowering the density by 20 % at the same grammage gives the same 1.953125——the two ratios differ by 0, so the same stiffness comes without more paper** (Section 3). **Fourth, raising the density softens it instead——at the same grammage, 1000 kg/m³ drops the stiffness to 0.512000** (Section 3). **Fifth, so grammage is not a unit of stiffness——at one grammage the stiffness ranges from 0.512000 to 1.953125** (Section 3). **Sixth, the separator is whether the quantity wanted is mass per unit area or bending stiffness——the unit one orders in and the property one wants do not line up** (Section 4). **Keywords**: grammage, bending stiffness, cube of thickness, bulk, density, the unit ordered against the property wanted.

Scope of this paper (scope note): **No new theorem or law is claimed——the definition of grammage, the proportionality of bending stiffness to the cube of thickness, and the greater stiffness of bulkier paper at the same grammage are all standard (Timoshenko 1930, Niskanen 1998).No precision is claimed for representative values——a modulus of 3.0 GPa and a density of 800 kg/m³ are order-of-magnitude values.Anisotropy is not treated——the difference in stiffness between machine and cross direction is not treated.Other properties are not treated——that bulkier paper changes smoothness, printability and opacity is not treated; only stiffness is considered.No real paper is claimed——grades and standards are not treated.Failure is not treated——buckling, creasing and changes with humidity are not treated.Relation to earlier papers**: **Paper 590 showed that a clip on a paper aeroplane does not shorten how far it glides, deriving 4.9896 g for an A4 sheet from a grammage of 80 g/m²——there grammage was an input for the mass, and the conclusion was that the mass does not enter the distance. Here the same grammage is converted to thickness and enters the stiffness as a cube. One quantity, two uses.**What is added is **carrying grammage through thickness to stiffness**, **showing the ratios to agree for more grammage and for lower density**, **putting the loss from higher density at 0.512000**, **showing the range of stiffness at one grammage**, and **placing the separator on mass against stiffness**.

1. Notation and the Question

Setting: **grammage is mass per unit area. Thickness is grammage over density, and bending stiffness per unit width is proportional to the cube of thickness.**

$$t = \frac{g}{\rho}, \qquad D = \frac{E t^3}{12}$$

g is the grammage, ρ the density and E the modulus, taken as 3.0 GPa.

Question: **what does grammage determine?**

2. More Grammage Stiffens as a Cube

Computation 2.1: **the thickness and bending stiffness per unit width were derived for each pair of grammage and density.**

grammage	density	thickness	bending stiffness
80 g/m²	800 kg/m³	0.100000 mm	2.500000×10^{-4} N m
100 g/m²	800 kg/m³	0.125000 mm	4.882812×10^{-4} N m
80 g/m²	640 kg/m³	0.125000 mm	4.882812×10^{-4} N m
80 g/m²	1000 kg/m³	0.080000 mm	1.280000×10^{-4} N m

Observation 2.2

25 % more grammage gives 1.953125 times the stiffness.It matches the cube of the thickness ratio 1.25 to within 10^{-12}——a quarter more paper returns nearly double the stiffness.

Observation 2.3

the cube comes from the geometry of the section.What resists bending are the fibres far from the neutral plane——thickening adds far fibres and lengthens their distance at once.

3. The Way That Uses No More Paper Works Just as Well

Observation 3.1

this is the core: the second and third rows have identical stiffness.Raising the grammage to 100 g/m² and leaving it at 80 g/m² while lowering the density by 20 % give the same thickness and the same stiffness, the two ratios differing by 0——one uses a quarter more paper and the other uses no more at all.

Observation 3.2

so an order for “stiffer paper” can be met two ways.Raise the grammage, or make it bulkier——judged on stiffness alone, the recipient cannot tell which was done.

Observation 3.3

the reverse holds too.At the same grammage, raising the density to 1000 kg/m³ drops the stiffness to 0.512000——compacted paper is limp at the same weight.

Observation 3.4

so grammage is not a unit of stiffness.At one 80 g/m² the stiffness ranges from 0.512000 to 1.953125——specifying the grammage does not specify the stiffness.

Observation 3.5

bulk is not free.Other properties such as smoothness and opacity move with it——this paper does not count them, but it is worth writing that stiffness alone does not settle the choice.

4. The Separator

Observation 4.1

the separator is whether the quantity wanted is mass per unit area or bending stiffness.The unit one orders in is the first; the property one wants is the second——they connect only through the density, so without naming the density the order is not determined.

Observation 4.2

beside Paper 590.Paper 590 showed that a clip on a paper aeroplane does not shorten how far it glides, and derived the mass from the grammage——there grammage was an input for mass, and the mass did not enter the distance. Here it is an input for thickness and enters the stiffness as a cube. One quantity, two uses.

Observation 4.3

what is not said.Anisotropy, smoothness, real grades and failure are not treated, and two representative values are held fixed——what is shown is that grammage does not determine stiffness.

5. Roll Call and Closing

1. **thickness 0.100000 mm** (Computation 2.1): **80 g/m² at 800 kg/m³.**
2. **stiffness 2.500000×10^{-4} N m** (Computation 2.1): **the reference.**
3. **4.882812×10^{-4} at 100 g/m²** (Computation 2.1): **using more paper.**
4. **4.882812×10^{-4} at 640 kg/m³** (Computation 2.1): **using no more.**
5. **both ratios 1.953125** (Observation 3.1): **the core.**
6. **the two differ by 0** (Observation 3.1): **indistinguishable.**
7. **matches the cube** (Observation 2.2): **within 10^{-12}.**
8. **0.512000 at 1000 kg/m³** (Observation 3.3): **compacted is limp.**
9. **a range at one grammage** (Observation 3.4): **stiffness unspecified.**
10. **mass or stiffness** (Observation 4.1): **the separator; Paper 590.**

Closing: **There are two ways to stiffen paper, one using more paper and one not, and in stiffness the two are exactly identical.Bending stiffness goes as the cube of thickness, so against 80 g/m² at 800 kg/m³, giving 0.100000 mm and 2.500000×10^{-4} N m, raising the grammage to 100 g/m² gives 0.125000 mm and 4.882812×10^{-4} N m, a factor of 1.953125 matching the cube of 1.25 to within 10^{-12}.Yet leaving the grammage alone and lowering the density by 20 % gives the same 0.125000 mm and the same factor of 1.953125, the two ratios differing by 0——one uses a quarter more paper and the other uses no more at all, and judged on stiffness alone the recipient cannot tell which was done.The reverse holds too: at the same grammage, raising the density to 1000 kg/m³ drops the stiffness to 0.512000 ——compacted paper is limp at the same weight.So grammage is not a unit of stiffness——at one 80 g/m² the stiffness ranges from 0.512000 to 1.953125, and specifying the grammage does not specify the stiffness.Bulk is not free, though: smoothness and opacity move with it, and stiffness alone does not settle the choice.The separator is whether the quantity wanted is mass per unit area or bending stiffness——the two connect only through the density.Placed among the earlier papers——in Paper 590 grammage was an input for mass that did not enter the distance; here it is an input for thickness and enters as a cube.To be honest——anisotropy, smoothness and real grades are not treated, and two representative values are held fixed.**

References

1. Niskanen, K. (1998). *Paper Physics*. Fapet Oy.
2. Timoshenko, S. (1930). *Strength of Materials*. Van Nostrand.
3. Yamagishi, Y. (2026). *A Clip on a Paper Aeroplane Does Not Shorten How Far It Glides* [Paper 590]. Zenodo. 10.5281/zenodo.23014382

— From thinking to syncing. —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)