# In Distillation, the Stages Needed for One More Nine of Purity Approach a Constant Set Only by Relative Volatility

— At total reflux with symmetric top and bottom purity, a relative volatility of **1.50** needs **22.665920** stages for purity **0.99**, **34.068306** for **0.999** and **45.430495** for **0.9999** — the separator is whether one counts purity itself or the number of nines — [Paper 761]

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*— From thinking to syncing. —*

**Abstract**

**In distillation, higher purity needs more stages. Going from 0.99 to 0.999 to 0.9999, one readily supposes each step needs ever more.The stages needed for one more nine of purity are counted by purity and by relative volatility.No new theorem or law is claimed. First, more nines need more stages——at relative volatility 1.50, 22.665920 for purity 0.99 and 45.430495 for 0.9999** (Section 2). **Second, yet the price of one nine falls rather than rises——from 11.827875 to 11.402386 and down to 11.357792, monotonically** (Section 2). **Third, and this is the core. The price of one nine approaches a constant independent of purity——the limit is $2\ln 10/\ln\alpha = 11.357747$ stages, within $4.439\times10^{-5}$ of the last increment** (Section 2). **Fourth, that price is set only by relative volatility——94.387266 at 1.05, 6.643856 at 2.00, 2.861353 stages at 5.00** (Section 3). **Fifth, stepping stage by stage gives the same——23 stages from 0.01 at the bottom to 0.99 at the top, the ceiling of the continuous 22.665920** (Section 3). **Sixth, the separator is whether one counts purity itself or the number of nines——"higher purity needs more stages" does not say how many until one says which is counted** (Section 4). **Keywords**: distillation, theoretical stages, relative volatility, Fenske equation, nines of purity, total reflux.

**Scope of this paper (scope note)**: **No new theorem or law is claimed——the Fenske equation for minimum theoretical stages at total reflux, the multiplication of two-component odds by the relative volatility at each equilibrium stage, and the logarithmic cost of raising purity are all known (Fenske 1932, McCabe et al. 2005).Only minimum stages——only the minimum at total reflux is considered; actual reflux ratio, stage efficiency and feed location are not treated.An ideal system——relative volatility is constant along the column; azeotropes and non-ideality are not treated.Symmetric purity——top purity and bottom impurity are set equal.No real column is claimed——particular mixtures and equipment are not treated; relative volatilities from 1.05 to 5.00 are for comparison.Relation to earlier papers**: **Paper 294 showed that separation improves only as the square root of the length——its scope note stated that what agrees is the structure of adding variances. There adding stages grew resolution only as a square root. Here each stage multiplies the odds by the relative volatility, and the number of nines grows in proportion to stages. Variances being added, or odds being multiplied.**What is added is **showing that the stages per added nine fall monotonically toward 11.357747**, **tabulating that price as set only by relative volatility at four values**, **confirming by stage-by-stage stepping that the count is the ceiling of the continuous value**, and **placing the separator on purity against the number of nines**.

## 1. Notation and the Question

**Setting**: **in two-component distillation let the relative volatility be $\alpha$. At total reflux, each stage up multiplies the odds of light to heavy component by $\alpha$. With top purity $x$ and bottom impurity $1-x$ set equal, the minimum stages $N$ and the limiting stages per added nine are as follows.**

$$\frac{y}{1-y} = \alpha \frac{x}{1-x}, \qquad N = \frac{2\ln\big(x/(1-x)\big)}{\ln \alpha}, \qquad \Delta N_{\infty} = \frac{2\ln 10}{\ln \alpha}$$

**A relative volatility of $\alpha = 1.50$ is the reference.**

**Question**: **does each nine cost more as purity rises?**

## 2. The Price of a Nine Approaches a Constant

**Computation 2.1**: **at relative volatility 1.50, minimum stages by purity and the stages per added nine were derived.**

| purity | minimum stages | stages for the added nine |
|---|---|---|
| 0.9 | 10.838045 | —— |
| 0.99 | 22.665920 | 11.827875 |
| 0.999 | 34.068306 | 11.402386 |
| 0.9999 | 45.430495 | 11.362189 |
| 0.99999 | 56.788687 | 11.358191 |
| 0.999999 | 68.146478 | 11.357792 |

**Observation 2.2**

more nines need more stages, but the price of one nine falls.It decreases monotonically from $11.827875$ to $11.357792$——the reading that each nine costs more at higher purity is wrong.

**Observation 2.3**

this is the core: the price of one nine approaches a constant independent of purity.The limit is $2\ln 10/\ln\alpha = 11.357747$ stages, within $4.439 \times 10^{-5}$ of the last increment——“constant” holds only asymptotically, and at low purity the step from $0.99$ to $0.999$ is slightly dearer.

**Observation 2.4**

so $100$ % needs infinitely many stages.Paying about $11.357747$ stages per nine, the number of nines grows only finitely——purity adds nines in proportion to stages, and the nines never run out.

## 3. The Price Is Set Only by Relative Volatility

**Computation 3.1**: for four relative volatilities, the limiting stages per nine and the minimum stages for purity $0.99$ were derived.

| relative volatility | per nine | for purity 0.99 |
|---|---|---|
| 1.05 | 94.387266 | 188.362549 |
| 1.50 | 11.357747 | 22.665920 |
| 2.00 | 6.643856 | 13.258713 |
| 5.00 | 2.861353 | 5.710217 |

> **Observation 3.2**
>
> the price of one nine is set only by relative volatility.94.387266 stages at 1.05, 2.861353 at 5.00——the closer the boiling points, the dearer each nine, by orders of magnitude.

Computation 3.3: equilibrium stages were stepped one by one, counting stages from 0.01 at the bottom to 0.99 at the top.

> **Observation 3.4**
>
> stepping took 23 stages, reaching 0.991256 at the last.This is the ceiling of the continuous value 22.665920 ——the formula counts the same thing as multiplying the odds stage by stage.

## 4. The Separator

> **Observation 4.1**
>
> the separator is whether one counts purity itself or the number of nines (the logarithm of impurity).By purity, 0.999 to 0.9999 looks a tiny step; by nines, it is the same step of 11.362189 stages——“higher purity needs more stages” does not say how many until one says which is counted.

**Observation 4.2**

**beside Paper 294.Paper 294 showed that adding stages improves resolution only as a square root, arising from variances being added——here each stage multiplies odds by the relative volatility, and nines grow in proportion to stages. Whether variances add or odds multiply, stages work differently.**

**Observation 4.3**

**what is not said.Reflux ratio, stage efficiency, feed location and non-ideality are not treated, and chromatographic purity is not computed——what is shown is that the price of a nine approaches a constant set only by relative volatility.**

## 5. Roll Call and Closing

1. **$22.665920$ stages for purity $0.99$** (Computation 2.1): **relative volatility $1.50$.**
2. **$45.430495$ for $0.9999$** (Computation 2.1): **more nines, more stages.**
3. **one nine $11.827875$** (Computation 2.1): **the first.**
4. **$11.357792$** (Computation 2.1): **falling.**
5. **limit $11.357747$** (Observation 2.3): **the core.**
6. **$94.387266$ at $1.05$** (Computation 3.1): **close boiling points.**
7. **$2.861353$ at $5.00$** (Computation 3.1): **distant boiling points.**
8. **$6.643856$ at $2.00$** (Computation 3.1): **in between.**
9. **$23$ stages stepping** (Observation 3.4): **matching the ceiling.**
10. **purity or nines** (Observation 4.1): **the separator; Paper 294.**

**Closing**: **In distillation, the stages needed for one more nine of purity approach a constant set only by relative volatility.At total reflux with relative volatility $1.50$, purity $0.99$ needs $22.665920$ stages and $0.9999$ needs $45.430495$, so more nines need more stages——yet the stages per added nine fall monotonically from $11.827875$ to $11.357792$ .They approach $2\ln 10/\ln\alpha = 11.357747$ stages, independent of purity and set only by relative volatility: $94.387266$ at $1.05$, $2.861353$ at $5.00$——a nine does not get dearer at higher purity; each costs almost the same, and $100$ % needs infinitely many stages.Stepping stage by stage from $0.01$ to $0.99$ took $23$ stages, the ceiling of the continuous value.The separator is whether one counts purity itself or the number of nines.Placed among the earlier papers——in Paper 294 variances added and resolution grew as a square root; here odds multiply and nines grow in proportion**

**to stages.To be honest——only minimum stages at total reflux are considered, and reflux ratio, stage efficiency and non-ideality are not treated.**

## References


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