The Average Gap Between Primes Is Right at ln x, but the Largest Gap Moves Ever Further from That Average

— Counting with a sieve up to one million, the ratio of the average gap $x/\pi(x)$ to $\ln x$ rises toward 1, from 0.861695 at $x = 10^3$ to 0.922092 at 10^6 — the separator is whether one counts the average gap or the longest gap — [Paper 769]

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— From thinking to syncing. —

Abstract

The prime number theorem says that near x primes appear roughly every $\ln x$. That is a statement about averages.The average gap and the longest gap are counted side by side as the range grows.No new theorem or law is claimed. First, the average gap approaches $\ln x$——the ratio is 0.861695 at 10^3 and 0.922092 at 10^6 (Section 2). **Second, the largest gap moves away from $\ln p$——ratios 2.946443, 3.928244, 6.953520, 8.697998** (Section 2). **Third, and this is the core. The better the average fits, the further the maximum strays——up to one million, the average is 0.922092 of $\ln x$ and the maximum 8.697998 times $\ln p$** (Section 2). **Fourth, the largest gap is updated in jumps——52, 72, 86, 96, 112, 114** (Section 3). **Fifth, runs without primes can be made as long as one likes——$10! + 2$ to $10! + 10$ and $12! + 2$ to $12! + 12$ are all composite** (Section 3). **Sixth, the separator is whether one counts the average gap or the longest gap——the prime number theorem sets the average and does not set the maximum** (Section 4). **Keywords**: prime gaps, prime number theorem, largest gap, average and extreme, prime-free runs, sieve of Eratosthenes.

Scope of this paper (scope note): **No new theorem or law is claimed——the prime number theorem, the sieve of Eratosthenes, arbitrarily long composite runs after factorials, and conjectures on the growth of the largest gap are all known (Hardy and Wright 2008, Cramér 1936).The range is limited——counting was done with a sieve up to one million; the largest gaps beyond are not claimed.Conjectures are not treated——the conjecture that the largest gap grows like $(\ln p)^2$ is only named.Nothing is proved——the prime number theorem is cited, not proved.Composite runs are checked by trial division——the numbers after $10!$ and $12!$ were divided one by one.Relation to earlier papers**: **Paper 344 showed that strength is a function of volume, not a constant of the material——there the weakest flaw set the strength, and larger volumes were weaker. Here the average prime gap follows $\ln x$ while the longest gap strays further from the average as the range widens. An extreme not set by the average moves with the range.**What is added is **tabulating average and largest gaps over four ranges**, **showing the average ratio approaching $\mathbf{0.922092}$ while the maximum ratio widens to $\mathbf{8.697998}$**, **listing the updates of the largest gap**, **checking the composite runs after factorials**, and **placing the separator on average against longest**.

1. Notation and the Question

Setting: **let $\pi(x)$ be the number of primes up to x. The average gap is $x/\pi(x)$, which the prime number theorem says approaches $\ln x$. The largest gap is the largest difference between adjacent primes up to x. $n!+2$ to $n!+n$ are composite because k divides $n!+k$.**

$$\pi(x) \sim \frac{x}{\ln x}, \qquad \bar{g}(x) = \frac{x}{\pi(x)}, \qquad k \mid n!+k \quad (2 \le k \le n)$$

Counting up to one million uses the sieve of Eratosthenes.

Question: **if primes appear roughly every $\ln x$, is the longest gap also roughly $\ln x$?**

2. The Average Fits, the Maximum Strays

Computation 2.1: **over four ranges, the number of primes, the average gap, $\ln x$ and the largest gap were derived.**

x	primes	average gap	average$/\ln x$	largest gap	largest$/\ln p$
10^3	168	5.952381	0.861695	20	2.946443
10^4	1229	8.136697	0.883431	36	3.928244
10^5	9592	10.425354	0.905535	72	6.953520
10^6	78498	12.739178	0.922092	114	8.697998

Observation 2.2

the average gap approaches $\ln x$.The ratio rises from 0.861695 to 0.922092, and $\pi(10^6) = 78498$ matches the known value——just as the prime number theorem says.

Observation 2.3

this is the core: the better the average fits, the further the maximum strays.The ratio of the largest gap to $\ln p$ widens from 2.946443 to 8.697998, and the largest gap up to one million is 114, after 492113——the law of averages does not hold down the longest gap.

3. Updates, and Gaps That Can Be Made

Computation 3.1: the points up to one million where the largest gap was updated were listed, and the runs after $10!$ and $12!$ were checked by trial division.

largest gap	prime just before
52	19609
72	31397
86	155921
96	360653
112	370261
114	492113

Observation 3.2

the largest gap is updated not smoothly but in jumps.The update from 96 to 112 occurs near 360653 and 370261, and then stalls for a while——the longest gap does not trace a smooth curve as the average does.

Observation 3.3

runs without primes can be made as long as one likes.The 9 numbers $10! + 2$ to $10! + 10$ and the 11 numbers $12! + 2$ to $12! + 12$ were all composite by trial division——no law puts a ceiling on the length of the gap.

4. The Separator

Observation 4.1

the separator is whether one counts the average gap or the longest gap.The prime number theorem sets the average, whose ratio approaches 1 at 0.922092; it does not set the maximum, whose ratio widens to 8.697998——"primes appear every $\ln x$" does not say how far it holds until one says whether it speaks of the average or the worst case.

Observation 4.2

beside Paper 344.Paper 344 showed the weakest flaw setting strength, with larger volumes being weaker ——here too, an extreme not set by the average strays from it as the range widens.

Observation 4.3

what is not said.Largest gaps beyond one million, conjectures on their growth and proofs of the prime number theorem are not treated——what is shown is that average and maximum appear to follow different laws.

5. Roll Call and Closing

1. $\pi(10^6) = 78498$ (Computation 2.1): **the known value.**
2. **average ratio 0.861695** (Computation 2.1): **at 10^3.**
3. **average ratio 0.922092** (Observation 2.2): **approaching 1.**
4. **largest ratio 2.946443** (Computation 2.1): **at 10^3.**

5. **largest ratio 8.697998** (Observation 2.3): **the core.**
6. **largest gap 114** (Computation 2.1): **after 492113.**
7. **average gap 12.739178** (Computation 2.1): **at 10^6.**
8. **update from 96 to 112** (Observation 3.2): **in jumps.**
9. **$12! + 2$ to $12! + 12$** (Observation 3.3): **a gap that can be made.**
10. **average or longest** (Observation 4.1): **the separator; Paper 344.**

Closing: **The average gap between primes is right at $\ln x$, but the largest gap moves ever further from that average.Counting with a sieve up to one million, the ratio of the average gap to $\ln x$ rises from 0.861695 at 10^3 to 0.922092 at 10^6, as the prime number theorem says——yet the ratio of the largest gap to $\ln p$ widens: 2.946443, 3.928244, 6.953520, 8.697998.The largest gap up to one million is 114 after 492113, updates from 52 to 114 come in jumps, and prime-free runs such as $10! + 2$ to $10! + 10$ or $12! + 2$ to $12! + 12$ can be made as long as one likes——the law of averages does not hold down the longest gap.The separator is whether one counts the average gap or the longest gap——the prime number theorem sets the average and does not set the maximum.Placed among the earlier papers——Paper 344 showed the weakest flaw lowering strength with volume; this counts, in the sequence of primes, an extreme straying from the average as the range widens.To be honest——computation stops at one million, and neither conjectures nor proofs are treated.**

References

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