

Liouville's Theorem and the Arrow of Time: Fine-Grained Conservation and Coarse-Grained Growth — Closing the Arc of Paper 75 on the Hamiltonian Hinge of Paper 76 — [Paper 77]

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— *From thinking to syncing.* —

ABSTRACT

Paper 76 placed, on phase space (the cotangent bundle T^*M), the spacetime metric (the base's $\sqrt{g_{00}}$) and the velocity-space curvature (the fiber's $E = 4\pi/|K|$) into a single "Hamiltonian hinge" via the canonical symplectic two-form $\omega = dp_\mu \wedge dx^\mu$ and the Hamiltonian $H(x, p)$. This paper treats the **next standard fact that appears when one turns a single page on that foundation — Liouville's theorem**. Under Hamiltonian flow ω is invariant ($\mathcal{L}_{X_H}\omega = 0$), and hence the phase-space volume $\Omega = \omega^n/n!$ is conserved. Here an **apparent contradiction** arises: this microscopic dynamics is **reversible and carries no arrow of time** — yet Paper 75 stated that "the entropy of the causal sphere grows (there is an arrow)." This paper **resolves (reconciles)** the tension within the standard, royal-road framework. The key is the distinction between **fine-grained and coarse-grained**. The fine-grained Gibbs entropy $S_G = -k \int \rho \ln \rho d\Omega$ is **exactly constant** under Hamiltonian flow (a consequence of Liouville). Hence **the arrow of time cannot reside in the microscopic law; it resides on the side of coarse-graining — the resolution of observation**. Paper 75's causal-sphere entropy $S(t) = \pi c^2 t^2 / l_P^2$ is a horizon (coarse-grained) quantity and does **not contradict** Liouville's microscopic conservation. In the Universe-OS lens: **the OS's memory is microscopically non-destructive (zero information dissipation), but our finite resolution (coarse-graining) generates the arrow** — though "OS" is a lens, not a testable claim. **Scope**: Liouville's theorem is a borrowed standard fact; this paper produces no new mathematics and no new predictions. It does **not solve the arrow-of-time problem nor the past hypothesis** (Penrose's open conjecture; Paper 75). Fine-grained conservation and coarse-grained growth are **distinct quantities and distinct matters; they are not fused** (distinct roots; Papers 58/48). This paper only offers a **structural keystone** joining Papers 75 and 76.

Positioning of this paper: This paper is a **structural interpretation** of established physical laws and mathematical theorems. It claims neither the discovery of new physical laws nor the resolution of open problems. Liouville's theorem, the Gibbs entropy, and coarse-graining, as used here, are all standard and established tools; this paper adds no new theorem to them. The words "Universe OS," "non-destructive memory," and "resolution" are **interpretive lenses and metaphors**, not testable claims; we state this explicitly at the outset.

1. Introduction: Toward the Volume the Hinge Preserves

Paper 76 placed, on the stage of the cotangent bundle T^*M , the base's $\sqrt{g_{00}}$ (gravitational redshift, position) and the fiber's $E = 4\pi/|K|$ (rest energy, velocity) into distinct-role co-inhabitation via the canonical symplectic two-form $\omega = dp_\mu \wedge dx^\mu$ and the Hamiltonian $H(x, p)$ (the Hamiltonian hinge). Once the stage (T^*M, ω) is set, the most natural next question is: **how does the "area/volume" of this stage behave under motion (Hamiltonian flow)?**

The answer is one of the cornerstones of classical mechanics, Liouville's theorem. And that answer immediately produces an **apparent tension** with Paper 75 (the arrow of time = the growth of causal-sphere entropy). This paper resolves the tension correctly within the known framework, joining Papers 75 and 76 within one structure.

2. Standard Fact: Liouville's Theorem

Standard Fact (Liouville's Theorem)

On (T^*M, ω) , along the Hamiltonian vector field X_H generated by H ($\iota_{X_H}\omega = dH$), the symplectic two-form is **invariant**:

$$\mathcal{L}_{X_H}\omega = d(\iota_{X_H}\omega) + \iota_{X_H}(d\omega) = d(dH) + 0 = 0.$$

Hence the Liouville volume form $\Omega = \omega^n/n!$ is also conserved under Hamiltonian flow ($\mathcal{L}_{X_H}\Omega = 0$).

Equivalently, the phase-space density $\rho(x, p, t)$ is constant along the flow and satisfies the Liouville equation

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \{\rho, H\} = 0$$

(Arnold; Abraham–Marsden).

The geometric meaning is plain. Any region of phase space, under Hamiltonian flow, **may be distorted in shape but never grows or shrinks in volume**. Like an incompressible fluid, the "ink" of phase space neither expands nor contracts; it only mixes. This is the most basic conservation law of the motion generated by Paper 76's hinge H .

3. The Apparent Contradiction: Reversible Micro vs Growing Macro

Liouville's theorem means the microscopic dynamics is **reversible and carries no arrow of time**. That the volume is conserved means information is lost neither toward the future nor toward the past — in OS language, "the memory is non-destructive."

Yet Paper 75 seemed to say the opposite: the causal (light) sphere $r = ct$ grows with time, and its Bekenstein–Hawking entropy $S(t) = \pi c^2 t^2 / l_P^2$ grows **monotonically** (there is an arrow).

Tension 3.1 (Liouville vs the Second Law)

The microscopic law (Liouville) says "phase-space volume = information is conserved, reversible, no arrow." The macroscopic law (the second law; Paper 75) says "entropy grows, irreversible, an arrow." **Do the two contradict?**

This is the classical question since Loschmidt (the reversibility paradox), not one this paper newly raises. But on this paper's stage — Paper 76's Hamiltonian hinge — the question can be posed in **precise** form.

4. Resolution (Royal Road): The Arrow Resides in the Coarse, Not the Fine

The standard answer lies in the distinction between **fine-grained and coarse-grained**. The decisive fact is the following.

Standard Fact (Conservation of the Fine-Grained Gibbs Entropy)

The fine-grained Gibbs entropy

$$S_G = -k \int \rho \ln \rho \, d\Omega$$

is **exactly constant** under Hamiltonian flow. This is because ρ is constant along the flow (Liouville) and the volume element $d\Omega$ is conserved, so the integral of $\rho \ln \rho$ is invariant.

The consequence is weighty. **The arrow of time can never emerge from the microscopic law**, for the fine-grained Gibbs entropy neither grows nor shrinks. Whence, then, the entropy increase? — from **coarse-graining**.

Observation 4.1 (The Home of the Arrow)

The "ink" of phase space, under Liouville flow, is stretched thread-like while conserving its volume, and **intermingles filamentarily** throughout the region. Viewed at finite resolution (finite cells), each cell is filled with a "mixture of ink and empty space," and the **effective occupied volume grows**. The coarse-grained entropy S_{coarse} thus increases. **The fine is conserved; the coarse grows. The arrow resides in the resolution of the viewer.**

This is the standard picture since Gibbs (1902): the arrow of the second law appears **only once one introduces an "information-discarding operation"** — coarse-graining, restriction to a subsystem, or interaction with an environment (open systems). This paper merely confirms this standard resolution on Paper 76's hinge.

5. The Universe-OS Lens: Non-Destructive Memory, and an Arrow Born of Resolution

Read through the system's lens (metaphor), this standard fact becomes the following — again stated explicitly to be **a lens, not a testable claim**.

- **Fine = the OS's memory is non-destructive.** Liouville is "zero information dissipation" = the non-destructiveness of thermodynamic garbage collection. The Universe OS loses nothing microscopically.
- **Coarse = our rendering resolution.** The arrow of time appears not because the Universe OS loses information, but because **the observer (we) can only coarse-grain at a finite pixel resolution**. The arrow is not a property "of the Universe OS" but a property "of the resolution of the side reading the rendering."

This reading does not deny Paper 75's "arrow of time = growth of causal-sphere entropy." Rather, it **locates which layer that arrow lives in**: Paper 75's arrow is not the arrow of the microscopic law, but the arrow of a coarse-grained quantity, the horizon.

6. Consistency with Paper 75: The Causal-Sphere Entropy Is a Coarse-Grained Quantity

Observation 6.1 (Paper 75 Does Not Contradict Liouville)

Paper 75's causal-sphere entropy $S(t) = \pi c^2 t^2 / l_P^2$ is a quantity proportional to the **area of the horizon** (Bekenstein–Hawking type), not the microscopic phase-space measure $d\Omega$ itself. Area-law entropy is essentially a **coarse-grained, boundary quantity** (it does not count the microscopic degrees of freedom beyond the horizon, but represents them by the boundary area). Hence its monotone growth is **not a statement at the same layer as, and does not contradict**, Liouville's microscopic volume conservation. The two are distinct quantities and distinct matters (distinct roots; Papers 58/48).

Thus Papers 75 (arrow) and 76 (hinge) are joined within one structure: **the microscopic flow generated by Paper 76's Hamiltonian hinge is reversible by Liouville (no arrow), while Paper 75's causal-sphere entropy grows as its coarse-grained face (an arrow).** They are two faces, at two resolutions — fine and coarse — of one stage (T^*M, ω) .

7. Scope (What Is Shown, and What Is Not)

What is shown (descriptive):

- Under the flow generated by Paper 76's hinge H , ω and the Liouville volume are conserved (standard fact).
- The fine-grained Gibbs entropy is exactly constant, so **the arrow of time resides in coarse-graining, not in the microscopic law** (Observation 4.1).
- Paper 75's causal-sphere entropy is a coarse-grained, boundary quantity and does **not contradict** Liouville's microscopic conservation (Observation 6.1), joining 75 and 76 within one structure.

What is not shown (restraint):

- **No new mathematics.** Liouville's theorem and the conservation of the fine-grained Gibbs entropy are entirely standard. This paper adds no theorem.
- **The arrow-of-time problem is not solved.** "Why was the past low-entropy?" (the past hypothesis) remains open (Penrose's conjecture; Paper 75). What this paper shows is the **home** of the arrow ("it resides in coarse-graining"), **not the reason for the low-entropy initial condition.**
- **Fine-grained conservation and coarse-grained growth are not fused.** They are distinct quantities at distinct layers (distinct roots; Papers 58/48). "The volume is conserved yet the entropy grows" is not a contradiction but **the distinct behaviour of two distinct quantities.**
- **No new prediction is made.** The framework yields the same observational results as standard statistical and Hamiltonian mechanics. Hence this paper remains at Layer 2 (structural interpretation).

8. Conclusion

On the Hamiltonian hinge (T^*M, ω, H) set by Paper 76, this paper confirmed its most basic conservation law — Liouville's theorem — and showed that it does **not contradict** the arrow of time of Paper 75. The key is the fine/coarse distinction: **the microscopic phase-space volume and the Gibbs entropy are exactly conserved (reversible, no arrow); the arrow of time resides on the side of coarse-graining — of finite resolution.** Paper 75's causal-sphere entropy is nothing but its coarse-grained, boundary face.

In the Universe-OS lens: **the OS's memory is microscopically non-destructive (zero information dissipation), but our finite resolution generates the arrow.** Yet "OS" and "resolution" are lenses, and what has been shown is merely an

observation that the correct structure joining Papers 75 and 76 lies within standard statistical mechanics. The deepest question — why the past was low-entropy — remains open. Even so, locating the "home" of the arrow in the coarse rather than the fine closes one more page of the arc, correctly.

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