Adding One Point Shortens the Total Length of a Network, and the 120-Degree Angle Decides Whether It Does

— The minimum spanning tree of an equilateral triangle of side **1** is **2.000000**; adding a point inside gives **1.732051**, shorter by a share of **0.133975**, with the three branches meeting at **120.000000** degrees each — the separator is whether any angle of the triangle is **120** degrees or more — [Paper 770]

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— From thinking to syncing. —

Abstract

The shortest network joining given points is readily thought to be built from lines between those points. But adding new points inside the network makes the total even shorter.How much shorter, and when not, are counted.No new theorem or law is claimed. First, adding a point shortens the total——for an equilateral triangle, from 2.000000 to 1.732051, a share 0.133975 shorter (Section 2). **Second, at the added point the three branches meet at 120 degrees ——three angles of 120.000000 degrees each** (Section 2). **Third, and this is the core. If an angle is 120 degrees or more, adding a point does not help——in a triangle with a 150 degree angle, the best point falls onto the vertex and the total stays 2.000000** (Section 3). **Fourth, allowed two points, a square shortens too——2.732051 against a minimum spanning tree of 3.000000, 0.910684 times** (Section 3). **Fifth, the position of those points is set by geometry too——the square's points sit at height 0.288675, equal to $1/\sqrt{12}$** (Section 3). **Sixth, the separator is whether any angle of the triangle is 120 degrees or more——"joined by the shortest network" does not say whether it can be shorter until one says whether points may be added and what the angles are** (Section 4). **Keywords**: Steiner point, minimum spanning tree, Steiner tree, branches at 120 degrees, total length, networks with added points.

Scope of this paper (scope note): **No new theorem or law is claimed——Steiner points, branches meeting at 120 degrees, the point falling onto a vertex when an angle is 120 degrees or more, the square's Steiner tree of length $1+\sqrt{3}$, and the Steiner ratio conjecture are all known (Gilbert and Pollak 1968, Courant and Robbins 1941).Small shapes only——only an equilateral triangle, a triangle with a 150 degree angle and a square are computed; Steiner trees for general point sets are not.Hardness is not treated——that finding Steiner trees is hard in general is only named.Conjectures are not treated——the conjecture on the largest possible saving is only named.No real design is claimed——piping, wiring and road design are not treated.Relation to earlier papers**: **Paper 348 showed that on the same graph, connecting is cheap and ordering is not——its scope note stated that weighted minimum spanning trees and approximation methods would not be treated, considering only how to connect the given points. Here, if points may be added, the network can be shorter than the minimum spanning tree. Connecting only the given points, or adding points to connect.**What is added is **finding the Steiner point by iteration and checking its angles and total, showing the added point falling onto the vertex in the 150 degree triangle, finding the square's two-point arrangement by search and checking it against $1+\sqrt{3}$,** and **placing the separator on the presence of an angle of 120 degrees or more**.

1. Notation and the Question

Setting: **join points in the plane with lines. The shortest network with lines only between the given points is the minimum spanning tree; the shortest one allowed to add new points is the Steiner tree. When three points are joined to one point s, the s minimising the total is found by the iteration below.**

$$L(s)=\sum_i \|s-p_i\|, \qquad s \leftarrow \frac{\sum_i p_i/\|s-p_i\|}{\sum_i 1/\|s-p_i\|}$$

For the square, two points are placed symmetrically, and the total $4\sqrt{1/4+t^2}+1-2t$ is minimised as a function of their height t.

Question: **can the shortest network joining given points be made shorter still?**

2. Adding a Point Shortens It

Computation 2.1: **for an equilateral triangle of side 1, the minimum spanning tree was compared with the network with one added point.**

quantity	value
minimum spanning tree	**2.000000**
network with one added point	**1.732051**
ratio	**0.866025**
share shorter	**0.133975**
added point	**(0.500000000, 0.288675135)**
angles between branches	**120.000000**, **120.000000**, **120.000000** degrees

> **Observation 2.2**
>
> **adding one point shortens the total from 2.000000 to 1.732051.The share saved is 0.133975, and the added point matches the centroid to 1.110×10^{-16}——a network using only the given points is not the shortest.**

> **Observation 2.3**
>
> **at the added point the three branches meet at 120 degrees.The largest deviation of the three angles is 1.421×10^{-14} degrees——because the directions in which three tensions balance are exactly 120 degrees apart.**

3. A Shape That Does Not Shorten, and a Two-Point Shape

Computation 3.1: **the same comparison was made for a triangle with a 150 degree angle and for a unit square.**

shape	minimum spanning tree	with added points	added points
150 degree triangle	**2.000000**	**2.000000**	vertex **(0.000000, 0.000000)**
square	**3.000000**	**2.732051**	two points, height **0.288675**

Observation 3.2

this is the core: if an angle is 120 degrees or more, adding a point does not shorten the network.In the 150 degree triangle the best point found by iteration fell onto the 150 degree vertex, and the total stayed 2.000000——at that vertex the branches are already spread wider than 120 degrees, so moving the point inside gains nothing.

Observation 3.3

allowed two points, the square shortens too.The total is 2.732051, differing from $1+\sqrt{3}$ by 0, which is 0.910684 times the minimum spanning tree; the height 0.288675 matches $1/\sqrt{12}$——here too the branches open at 120 degrees at the two added points.

4. The Separator

Observation 4.1

the separator is whether any angle of the triangle is 120 degrees or more; if so, the added point falls onto the vertex and nothing is shortened.With the equilateral triangle's 60 degree angles the saving is 0.133975; with a 150 degree angle it is 0——"joined by the shortest network" does not say whether it can be shorter until one says whether points may be added and what the angles are.

Observation 4.2

beside Paper 348.Paper 348 showed that the minimum spanning tree joining given points is cheap to find while an ordering tour is not——here, if points may be added, the network can be even shorter than that cheap minimum spanning tree.

Observation 4.3

what is not said.Steiner trees for general point sets, their hardness, conjectures on the saving and real wiring are not treated——what is shown is that adding points shortens a network, and that a 120 degree angle stops it.

5. Roll Call and Closing

1. **minimum spanning tree 2.000000** (Computation 2.1): **equilateral triangle.**
2. **with a point 1.732051** (Computation 2.1): **shorter.**
3. **share 0.133975** (Observation 2.2): **by how much.**
4. **ratio 0.866025** (Computation 2.1): **$\sqrt{3}/2$.**
5. **angles 120.000000 degrees** (Observation 2.3): **balanced tensions.**
6. **2.000000 unchanged at 150 degrees** (Observation 3.2): **the core.**
7. **square 2.732051** (Computation 3.1): **two points added.**
8. **ratio 0.910684** (Observation 3.3): **the square's saving.**
9. **height 0.288675** (Observation 3.3): **$1/\sqrt{12}$.**
10. **an angle of 120 degrees or more** (Observation 4.1): **the separator; Paper 348.**

Closing: **Adding one point shortens the total length of a network, and the 120-degree angle decides whether it does.The minimum spanning tree of an equilateral triangle of side 1 is 2.000000, but adding a point inside gives 1.732051, a share 0.133975 shorter, with three branches meeting at 120.000000 degrees each——a network using only the given points is not the shortest.Yet in a triangle with a 150 degree angle the best point falls onto the vertex and the total stays 2.000000: nothing is shortened——at that vertex the branches already spread wider than 120 degrees.Allowed two points, a unit square also shortens from 3.000000 to 2.732051 (0.910684 times).The separator is whether any angle of the triangle is 120 degrees or more——"joined by the shortest network" does not say whether it can be shorter until one says whether points may be added and what the angles are.Placed among the earlier papers——Paper 348 treated minimum spanning trees on given points; this counts how networks allowed extra points become shorter still.To be honest——only three small shapes are computed, not general point sets or real wiring.**

References

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