The Number of Directions You Can Push Is Not the Dimension of Where You Can Go

— going forward, turning, backing and turning back by ε each moves it in the unpushed sideways direction, with sideways displacement over ε^2 at -0.998334 for $\varepsilon = 0.1$ and -1.000000 for 0.001 — the separator is whether the discrepancy from alternating two pushes points outside the directions that can be pushed — [Paper 777]

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— From thinking to syncing. —

Abstract

A car cannot move sideways. It can only drive forward or back and turn. Yet it can pull into a parking space beside it.How a third direction becomes reachable with only two directions of push is counted.No new theorem or law is claimed. First, two directions to push, three dimensions to reach——three states, position and heading; two inputs, forward and turning (Section 2). **Second, alternating pushes moves the car in an unpushed direction——over one loop of forward, turn, back, turn back, sideways displacement over ε^2 goes from -0.998334 to -1.000000** (Section 2). **Third, the Lie bracket sets that direction——the numerical bracket $(0.644218, -0.764842, 0)$ matches the formula to 2.184×10^{-11}** (Section 3). **Fourth, and this is the core. If the bracket points outside the pushable directions, every direction is reachable——the determinant across three directions is 1.000000** (Section 3). **Fifth, if the bracket is 0, some direction stays out of reach——a cart pushed only in x and y has bracket 0 and cannot reach other headings** (Section 3). **Sixth, the separator is whether the discrepancy from alternating two pushes (the Lie bracket) points outside the directions that can be pushed——"only two controls, so only two dimensions of motion" has no truth value until the bracket is stated** (Section 4). **Keywords**: underactuated systems, non-holonomic constraints, Lie bracket, reachability, car kinematics, parallel parking.

Scope of this paper (scope note): **No new theorem or law is claimed——the kinematic car model, non-holonomic constraints, the Lie bracket, reachability when brackets span all dimensions (Chow's theorem), and sideways motion by small back-and-forth manoeuvres are all known (Murray et al. 1994, LaValle 2006).Kinematics only——mass, inertia, slip and tyre forces are not treated; only a velocity-driven model is used.No theorem is proved——Chow's theorem and higher brackets are only named.No procedure is recommended——real parallel-parking procedures are not treated.Checked numerically——the loop was integrated by the midpoint method, and the bracket found by finite differences.Relation to earlier papers**: **Paper 601 showed that half of the fifteen puzzle's positions can never be reached——there an invariant attached to the moves put half the positions out of reach. Here, with only two directions of push, the discrepancy of alternating pushes makes a third direction and everything is reached. An invariant that makes an unreachable side, and a bracket that enlarges the reachable side.**What is added is **integrating the loop's displacement at three values of ε and showing it proportional to ε^2, checking the numerical Lie bracket against the formula and deriving a determinant of $\mathbf{1.000000}$, comparing with a cart whose bracket is 0**, and **placing the separator on whether the bracket points outside the pushable directions**.

1. Notation and the Question

Setting: **let the car's state be position (x, y) and heading θ, with two inputs, forward speed v and turning rate ω. The discrepancy between the two input directions g_1 and g_2 is expressed by the Lie bracket.**

$$\dot{\mathbf{q}} = v\,g_1 + \omega\,g_2, \qquad g_1 = (\cos\theta,\ \sin\theta,\ 0), \quad g_2 = (0,\ 0,\ 1), \qquad [g_1, g_2] = (\sin\theta,\ -\cos\theta,\ 0)$$

Going forward, turning left, backing and turning right by ε each displaces the car by $\varepsilon^2[g_1, g_2]$.

Question: **can a car with only two directions of push reach anywhere in three dimensions?**

2. Alternating Pushes Move It in an Unpushed Direction

Computation 2.1: **the loop was integrated from heading $\theta = 0$, and the displacement in position and heading derived.**

ε	forward displacement	sideways displacement	sideways$/\varepsilon^2$
0.100	4.995835×10^{-4}	-9.983342×10^{-3}	-0.998334
0.010	4.999958×10^{-7}	-9.999833×10^{-5}	-0.999983
0.001	4.999999×10^{-10}	-9.999998×10^{-7}	-1.000000

Observation 2.2

alternating pushes moves the car sideways, a direction never pushed.Sideways displacement is proportional to ε^2, with sideways over ε^2 at -1.000000 for $\varepsilon = 0.001$, while heading returns to within 1.857×10^{-18}——nothing pushes sideways, yet the loop's discrepancy carries the car there.

Observation 2.3

large sideways moves take many small back-and-forth loops.Moving 0.01 m sideways needs $\varepsilon = 0.100000$ per loop, with 0.200000 m of forward and backward travel——since the displacement is ε^2, a sideways step is much smaller than the forward steps.

3. The Bracket Sets the Direction

Computation 3.1: **at heading $\theta = 0.7$, the Lie bracket was found by finite differences and compared with the formula, and the determinant of g_1, g_2 and the bracket derived. The bracket of a cart pushed only in x and y was also derived.**

quantity	value
Lie bracket (numerical)	$(0.644218, -0.764842, 0.000000)$
formula $(\sin\theta, -\cos\theta, 0)$	$(0.644218, -0.764842, 0.000000)$
largest difference	2.184×10^{-11}
determinant across three directions	1.000000
bracket of the cart that cannot turn	$(0, 0, 0)$

Observation 3.2

this is the core: if the bracket points outside the pushable directions, every direction is reachable.The car's bracket points sideways, and with g_1 and g_2 its determinant is 1.000000: three directions are spanned——two directions of push, three dimensions of reach.

Observation 3.3

if the bracket is 0, a direction stays out of reach.A cart pushed only in x and y shows no discrepancy however the pushes alternate, and its heading θ never changes——what decides the reachable dimension is not the number of pushes but whether they fail to commute.

4. The Separator

Observation 4.1

the separator is whether the discrepancy from alternating two pushes (the Lie bracket) points outside the directions that can be pushed.If it does, everything is reachable, as for the car; if it is 0, some direction stays unreachable, as for the cart——"only two controls, so only two dimensions of motion" has no truth value until the bracket is stated.

Observation 4.2

beside Paper 601.Paper 601 showed an invariant attached to the fifteen puzzle's moves putting half its positions out of reach——there a constraint cut the reachable set; here, even under a constraint, the discrepancy of alternating pushes enlarges the reachable directions.

Observation 4.3

what is not said.Mass, slip, proofs of theorems and real parking procedures are not treated——what is shown is that the number of pushes and the dimension of reach differ, and that the bracket separates them.

5. Roll Call and Closing

1. **two inputs, three states** (Section 1): **underactuated.**
2. $\mathbf{-0.998334}$ **at** $\varepsilon = 0.1$ (Computation 2.1): **moving sideways.**
3. $\mathbf{-1.000000}$ **at** $\varepsilon = 0.001$ (Observation 2.2): **proportional to** ε^2**.**
4. **heading error** 1.857×10^{-18} (Observation 2.2): **heading returns.**

5. $\varepsilon = 0.100000$ **for** 0.01 **m sideways** (Observation 2.3): **small loops.**
6. **bracket** $(0.644218, -0.764842, 0)$ (Computation 3.1): **matching the formula.**
7. **determinant** 1.000000 (Observation 3.2): **the core.**
8. **cart bracket** 0 (Observation 3.3): **heading out of reach.**
9. 0.200000 **m of forward and back travel** (Observation 2.3): **the price of a sideways step.**
10. **bracket outside the pushable directions or not** (Observation 4.1): **the separator; Paper 601.**

Closing: **The number of directions you can push is not the dimension of where you can go.A car model has three states, position and heading, but only two inputs, forward motion and turning——yet one loop of forward, turn, back and turn back moves it sideways, a direction never pushed, with sideways displacement over ε^2 approaching -1.000000 at $\varepsilon = 0.001$ from -0.998334 at 0.1.That direction is the Lie bracket, whose numerical value $(0.644218, -0.764842, 0)$ matches the formula to 2.184×10^{-11}, and with g_1 and g_2 gives a determinant of 1.000000 , spanning all three directions; a cart that cannot turn has bracket 0 and cannot reach other headings——the reachable dimension is set not by the number of pushes but by whether they fail to commute.The separator is whether the discrepancy from alternating two pushes (the Lie bracket) points outside the directions that can be pushed.Placed among the earlier papers——Paper 601 showed an invariant cutting the reachable positions in half; this counts a bracket enlarging the reachable directions.To be honest——only a kinematic model is used, with no mass, slip, proofs or real parking.**

References

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