A Two-Link Arm Reaches a Target in 0, 1 or 2 Ways by Where It Is, and One More Joint Makes a Continuous Range

— targets at 0.1, 0.2, 0.5, 0.8 and 0.9 m are reached by 0, 1, 2, 1 and 0 joint-angle sets, and three joints give an unbroken 1.664695 rad — the separator is whether the target is inside, on the edge of or outside the reachable region, and whether the joints outnumber its degrees of freedom — [Paper 791]

Yuuki YAMAGISHI (山岸 勇輝)
Yamagishi Complexity Torus Lab, Japan
September 2026

— *From thinking to syncing.* —

Abstract

Asking whether a robot arm can reach a place, the answer seems to be one of two: it can or it cannot.With planar arms of two and three links, how many joint-angle sets reach each target position, and the price paid near the edge, are counted.No new theorem or law is claimed. First, inside the reachable region there are two ways to reach——2 ways at 0.5 m (elbow up and elbow down) (Section 2). **Second, exactly one way on the edge, none outside——1 way at 0.2 and 0.8 m, 0 ways at 0.1 and 0.9 m; Newton's method finds the same counts** (Section 2). **Third, and this is the core. The closer to the edge, the more the price of reaching jumps——the joint speed for moving the hand outward at 0.1 m/s is 0.568591 rad/s at $r = 0.7$ m and 17.440886 rad/s at 0.7999 m** (Section 3). **Fourth, the reachable region is a ring——area 1.884956 m^2; random sampling gives 0.736870 (formula 0.736311)** (Section 2). **Fifth, one more joint and the solutions are no longer a countable number——the first joint angle reaching the same target runs unbroken over 1.664695 rad (0.264944 of the full turn)** (Section 4). **Sixth, the separator is whether the target is inside, on the edge of or outside the reachable region, and whether the number of joints exceeds the target's degrees of freedom——"can this arm reach that place" is not answered until one says in how many ways and at what price at the edge** (Section 5). **Keywords**: inverse kinematics, robot arm, singular configuration, workspace, redundancy, Jacobian.

Scope of this paper (scope note): **No new theorem or law is claimed——inverse kinematics of a two-joint planar arm, the two elbow-up and elbow-down solutions, the merging of solutions and vanishing Jacobian at the workspace edge (singular configurations), and continuous solution sets when joints outnumber the target's degrees of freedom (redundancy) are all known (Craig 2005, Siciliano et al. 2009).Kinematics only——forces, masses, dynamics, joint limits and obstacles are left out.Planar only——three-dimensional arms and hand orientation are not treated.Dimensions are representative——links of 0.5 m and 0.3 m, and 0.4, 0.25 and 0.15 m for three joints, are for comparison.Control is not treated——which solution a real robot chooses is not stated.Relation to earlier papers**: **Paper 236 showed that one and the same "= 0" returns five different kinds of collection——its scope note stated that every section only counts known solutions and proves no existence theorem. There the kind of solution set split by which number world one solves in. Here, for the same arm equation, the number of solutions is counted as splitting into 0, 1, 2 and an unbroken range by target position and number of joints.**What is added is **counting solutions at five distances two ways, by formula and by Newton's method**, **deriving the Jacobian and joint speeds near the edge**, **deriving the reachable area by formula and by random sampling**, **deriving the width of solutions with three joints**, and **placing the separator on inside, edge or outside, and on the number of joints**.

1. Notation and the Question

Setting: **let a planar arm with links L_1 and L_2 have joint angles θ_1 and θ_2. The hand position, the elbow angle from the target distance r, and the Jacobian determinant are as follows.**

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2), \qquad y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

$$\cos\theta_2 = \frac{r^2 - L_1^2 - L_2^2}{2 L_1 L_2}, \qquad \det J = L_1 L_2 \sin\theta_2$$

If the right-hand side has absolute value below 1, θ_2 has two values of opposite sign; exactly 1 gives one; above 1, none. Take $L_1 = 0.5$ m and $L_2 = 0.3$ m.

Question: **does the arm reach the target, and if so in how many ways and at what price?**

2. Number of Ways to Reach

Computation 2.1: **for targets at five distances, solutions were derived by formula and compared with the number found by Newton's method from 144 initial guesses.**

target distance	formula solutions	Newton solutions	hand error (max)
0.1 m	0	0	—
0.2 m	1	1	6.94×10^{-18} m
0.5 m	2	2	7.85×10^{-17} m
0.8 m	1	1	5.55×10^{-17} m
0.9 m	0	0	—

Observation 2.2

two ways inside the reachable region, one on the edge, none outside.At 0.5 m there are elbow-up and elbow-down forms; on the inner edge 0.2 m and outer edge 0.8 m the two merge into one; 0.1 m and 0.9 m cannot be reached——"reaching" is not binary but has three answers, 0, 1 and 2.

Observation 2.3

the reachable region is a ring.Its area is $\pi\{(L_1+L_2)^2-(L_1-L_2)^2\} = 1.884956$ m^2, 0.736311 of a 1.6 m square, and random sampling gave 0.736870.

3. The Price at the Edge

Computation 3.1: **near the outer edge, the Jacobian determinant and the joint speed needed to move the hand outward at 0.1 m/s were derived.**

target distance	Jacobian determinant	joint speed
0.7 m	1.299038×10^{-1}	0.568591 rad/s
0.79 m	4.818503×10^{-2}	1.749174 rad/s
0.799 m	1.546644×10^{-2}	5.516748 rad/s
0.7999 m	4.898173×10^{-3}	17.440886 rad/s

Observation 3.2

this is the core: the closer to the edge, the more the price of reaching jumps.Approaching the outer edge to within 0.1, 0.01, 0.001 and 0.0001 m, the Jacobian determinant shrinks from 1.299038×10^{-1} to 4.898173×10^{-3}, and the joint speed needed for the same hand speed keeps rising from 0.568591 to 17.440886 rad/s——at the edge the two solutions merge into one, and just before it a slight hand motion demands a large joint motion.

4. One More Joint

Computation 4.1: with three joints (0.4, 0.25 and 0.15 m), the first joint angle was stepped through 36000 values to reach the target $(0.5, 0.2)$, counting the width of angles for which the remaining two links reach it.

Observation 4.2

one more joint and the solutions are no longer a countable number.The first joint angle runs unbroken over 1.664695 rad (0.264944 of the full turn), and anywhere in that range the remaining two links reach the target——once joints outnumber the target's degrees of freedom (the 2 of a planar position), the solutions become an unbroken range rather than 0, 1 or 2.

5. The Separator

Observation 5.1

the separator is whether the target is inside, on the edge of or outside the reachable region, and whether the number of joints exceeds the target's degrees of freedom.With two links there are 2 ways inside, 1 on the edge, 0 outside, and near the edge the joint speed jumps to 17.440886 rad/s; with three, the solutions form an unbroken range of 1.664695 rad——"can this arm reach that place" is not answered until one says in how many ways and at what price at the edge.

Observation 5.2

beside Paper 236.Paper 236 showed that one and the same "$= 0$" returns five kinds of solution set by which number world one solves in——here, for the same arm equation, the number of solutions is counted as splitting into 0, 1, 2 and an unbroken range by target position and number of joints.

Observation 5.3

what is not said.Neither forces and masses, joint limits, obstacles, three-dimensional arms nor the control of real robots is treated——what is shown is that the question of reach carries a number of solutions and a price at the edge.

6. Roll Call and Closing

1. **2 ways at 0.5 m** (Computation 2.1): **elbow up and down.**
2. **1 way on the edges 0.2 and 0.8 m** (Observation 2.2): **the two merge.**
3. **0 ways at 0.1 and 0.9 m** (Observation 2.2): **out of reach.**
4. **reachable area 1.884956 m^2** (Observation 2.3): **a ring.**
5. **0.568591 rad/s at $r = 0.7$ m** (Computation 3.1): **far from the edge.**
6. **17.440886 rad/s at $r = 0.7999$ m** (Observation 3.2): **the core.**
7. **determinant 4.898173×10^{-3}** (Computation 3.1): **nearing zero at the edge.**
8. **width 1.664695 rad with three joints** (Observation 4.2): **an unbroken solution.**
9. **0.264944 of a full turn** (Observation 4.2): **the share of the range.**
10. **inside, edge or outside; number of joints** (Observation 5.1): **the separator; Paper 236.**

Closing: **The same two-link arm reaches a target in 0, 1 or 2 ways depending on where it is, and one more joint turns this into a continuous range.For a planar arm with links of 0.5 m and 0.3 m, targets at 0.1, 0.2, 0.5, 0.8 and 0.9 m are reached by 0, 1, 2, 1 and 0 angle sets, the same counts found by Newton's method, within a ring of area 1.884956 m^2——"reaching" has three answers, not two.Near the outer edge the joint speed for moving the hand outward at 0.1 m/s jumps from 0.568591 to 17.440886 rad/s, and with three joints the angles reaching the same target run unbroken over 1.664695 rad——at the edge solutions merge and the price jumps; when joints exceed the target's degrees of freedom, solutions become a range.The separator is whether the target is inside, on the edge of or outside the reachable region, and whether the number of joints exceeds the target's degrees of freedom.Placed among the**

earlier papers——Paper 236 counted kinds of solution set; this counts the number of solutions of one equation changing with target position and number of joints.To be honest——it is planar kinematics only, with no forces, joint limits or real control.

References

1. Craig, J. J. (2005). *Introduction to Robotics: Mechanics and Control* (3rd ed.). Pearson Prentice Hall.
2. Siciliano, B., Sciavicco, L., Villani, L., & Oriolo, G. (2009). *Robotics: Modelling, Planning and Control*. Springer.
3. Yamagishi, Y. (2026). *One and the Same "= 0" Returns Five Different Kinds of Collection* [Paper 236]. Zenodo. 10.5281/zenodo.22129043

— From thinking to syncing. —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)