

A Structural Interpretation of the 4D Poincaré Conjecture and Exotic Structures via the $n=3$ Uniqueness Theorem and Hopf Fibration

— $2\pi^2$ Hyperspherical Unit Mass Patch, $16\pi^2$ Toroidal Cocoon, and the Information-Geometric Boundary of

Exotic $\mathbb{R}^4$ — [Paper 08]

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September 2026 (4th Edition)

— *From thinking to syncing.* —

Abstract

For the four-dimensional topological Poincaré conjecture (Freedman's theorem) and the exotic differentiable structures of four-dimensional Euclidean space (exotic $\mathbb{R}^4$), we present a structural interpretation based on the $n = 3$ uniqueness theorem and the Hopf fibration. The argument proceeds in three stages.

Stage one: in $\mathbb{R}^4$, the shell mass growth rate $dm/dr = 2\pi^2 r^{3-n}$ of a density $\rho(r) = r^{-n}$ is constant only for $n = 3$, and its value is the three-sphere volume $\text{Vol}(S^3) = 2\pi^2$ (Theorem 2.1).

Stage two: for the Hopf fibration $S^1 \hookrightarrow S_R^3 \rightarrow S_{R/2}^2$ of the three-sphere of radius R , the product of the base area πR^2 and the fibre circumference $2\pi R$ equals the total space volume $2\pi^2 R^3$ for every R (Theorem 3.1). At $R = 2$ both factors take the numerical value 4π , and $16\pi^2 = \text{Vol}(S_{R=2}^3) = 4\pi \times 4\pi$ is realised as an actual geometric decomposition (Theorem 3.3). The $16\pi^2$ toroidal cocoon is thereby fixed, before it is the algebraic product $(4\pi)^2$, as **a single geometric object: the three-sphere of radius two**. Since the agreement of the two factors compares an area $[L^2]$ with a length $[L]$, rescaling the unit of length by λ moves the radius of agreement to $R = 2/\lambda$. What does not depend on units is the ratio of quantities of the same dimension, $\text{Vol}(S_{R=2}^3)/\text{Vol}(S_{R=1}^3) = 2^3 = 8$ (Corollary 3.4).

Stage three: the fact that uncountably many exotic $\mathbb{R}^4$ exist in four dimensions alone is read, in the language of information geometry, as the passage from the solid angle 4π to $16\pi^2$ (Section 4).

Scope of this paper (scope note): This paper presents a **structural interpretation** of established results in four-dimensional topology. The four-dimensional topological Poincaré conjecture was settled by Freedman (1982); this paper only cites it and takes no part in its proof. The smooth four-dimensional Poincaré conjecture remains open. The existence of exotic $\mathbb{R}^4$ follows from Freedman’s topological classification together with Donaldson’s gauge theory; Gompf (1985) constructed an infinite family, and Taubes (1987) showed that there are uncountably many. The reading of Section 4 is not a derivation of these results, and whether a rigorous correspondence exists between the volume factor $2^3 = 8$ and the theory of intersection forms is an open question. The decomposition of Theorem 3.1 holds as a standard instance of the volume formula for a Riemannian submersion with totally geodesic fibres (when the fibre length is constant, the volume of the total space is the volume of the base times the fibre length). **What this paper adds** is that it fixes $2\pi^2$ as the four-dimensional unit patch by the $n = 3$ uniqueness theorem and unpacks it into the Hopf fibre structure; that it gives the recurring quantity $16\pi^2$ a substance, **the three-sphere of radius two**, inside which base and fibre split into 4π each; and that it tells apart what moves with the choice of units — the radius at which that split becomes visible — from what does not, the volume ratio 8.

1. Introduction: The Singularity of 4D Topology and the Poincaré Conjecture

1.1 The 4D Poincaré Conjecture and Freedman’s Theorem

Michael Freedman (1982) settled the four-dimensional Poincaré conjecture in the topological category.

1.2 The Singularity of Differential Topology in Four Dimensions (Exotic $\mathbb{R}^4$)

Combining Freedman’s topological classification with Simon Donaldson’s gauge theory (1983), it follows that four-dimensional Euclidean space $\mathbb{R}^4$ carries differentiable structures not diffeomorphic to the standard one (exotic $\mathbb{R}^4$). Robert Gompf (1985) constructed an infinite family of them, and Clifford Taubes (1987) showed that there are uncountably many. For $n \neq 4$ the space $\mathbb{R}^n$ has only one differentiable structure; this richness is peculiar to four dimensions.

2. The $n = 3$ Uniqueness Theorem in Four-Dimensional Space

Theorem 2.1 ($n = 3$ Uniqueness Theorem — 4D Hyperspherical Patch Theorem)

In $\mathbb{R}^4$, for a spherically symmetric power-law density $\rho(r) = r^{-n}$, the mass growth rate of the three-dimensional hyperspherical shell S_r^3 ,

$$\frac{dm^{(4)}}{dr} = \text{Vol}(S^3) \cdot r^{3-n} = 2\pi^2 r^{3-n}$$

is uniform and constant on $(0, \infty)$ if and only if $n = 3$. The constant value is then the three-sphere volume $\text{Vol}(S^3) = 2\pi^2$.

Proof.

The volume element in $\mathbb{R}^4$ is $dV = 2\pi^2 r^3 dr$. The shell mass is $dm = \rho(r)dV = 2\pi^2 r^{3-n} dr$. The necessary and sufficient condition for $dm/dr = 2\pi^2 r^{3-n}$ to be independent of r is $3 - n = 0 \iff n = 3$, whereupon $dm/dr = 2\pi^2$.

This carries Theorem 1 of Paper 01 ($n = 2$ in three dimensions) over to four dimensions.

3. Geometric Unpacking via the Hopf Fibration

We decompose the four-dimensional unit patch $2\pi^2$ fixed in the previous section into the fibre structure of the Hopf fibration. In this section **the radius R of the three-sphere is treated as an explicit variable**. As we shall see, making the radius a variable is the key to telling what $16\pi^2$ really is.

Theorem 3.1 (Hopf Decomposition Theorem)

For the Hopf fibration of the three-sphere of radius R ,

$$S^1 \hookrightarrow S_R^3 \longrightarrow S_{R/2}^2$$

the product of the area of the base $S_{R/2}^2$ and the circumference of the fibre S^1 equals the volume of the total space for every $R > 0$:

$$\underbrace{\pi R^2}_{\text{base area}} \times \underbrace{2\pi R}_{\text{fibre circumference}} = 2\pi^2 R^3 = \text{Vol}(S_R^3)$$

Proof.

In the Hopf fibration, the three-sphere of radius R has as base the two-sphere of radius $R/2$. The base area is therefore

$$4\pi\left(\frac{R}{2}\right)^2 = \pi R^2$$

and the fibre, being a great circle of radius R , has circumference $2\pi R$. Their product is

$$\pi R^2 \cdot 2\pi R = 2\pi^2 R^3$$

which equals the volume $\text{Vol}(S_R^3) = 2\pi^2 R^3$ of the three-sphere of radius R .

Corollary 3.2 (The Unit Radius Case — the 4D Mass Patch)

At $R = 1$, Theorem 3.1 gives

$$2\pi^2 = \pi \times 2\pi$$

That is, the mass patch $2\pi^2$ fixed by the $n = 3$ uniqueness theorem unpacks as the product of base area π and fibre circumference 2π .

Theorem 3.3 (Twofold Appearance of the Yamagishi Patch at $R = 2$ — the Identity of the $16\pi^2$ Cocoon)

At $R = 2$, both factors of the Hopf decomposition coincide with the Yamagishi patch 4π , and the total space volume is $16\pi^2$:

$$16\pi^2 = \text{Vol}(S_{R=2}^3) = \underbrace{4\pi}_{\text{solid angle of base } S_{R/2=1}^2} \times \underbrace{4\pi}_{\text{fibre circumference}}$$

Moreover, under the standard metric, $R = 2$ is the only radius at which the **numerical values** of the two factors agree.

Proof.

Substituting $R = 2$ into Theorem 3.1: the base area is $\pi \cdot 2^2 = 4\pi$, the fibre circumference is $2\pi \cdot 2 = 4\pi$, and the total volume is $2\pi^2 \cdot 2^3 = 16\pi^2$.

For uniqueness, solve the coincidence condition

$$\pi R^2 = 2\pi R$$

for $R > 0$. Dividing both sides by $\pi R (\neq 0)$ gives $R = 2$.

3.1 The Radius of Agreement Moves with Units; the Volume Ratio Does Not

The two quantities compared in the agreement of Theorem 3.3 differ in dimension: the base area πR^2 is $[L^2]$ and the fibre circumference $2\pi R$ is $[L]$. The radius at which they agree therefore depends on the choice of the unit of length; rescaling the unit by λ moves it to

$$\pi(\lambda R)^2 = 2\pi(\lambda R) \implies R = \frac{2}{\lambda}$$

($R = 1$ for $\lambda = 2$; $R = 1/5$ for $\lambda = 10$). $R = 2$ is the radius chosen so that 4π appears twice. The discipline of not equating an area with a length is one Paper 18 established for an identification of the same kind (the circumference 2π of S^1 against the solid angle 4π).

By contrast, the following three are dimensionally consistent and hold independently of the choice of units.

- i. **Theorem 3.1 (the Hopf decomposition) itself:** $[L^2] \times [L] = [L^3]$ is dimensionally consistent, and it is a consequence of the volume formula for a Riemannian submersion with totally geodesic fibres.
- ii. $16\pi^2 = \text{Vol}(S_{R=2}^3)$: the arithmetic $2\pi^2 \cdot 2^3$; $16\pi^2$ is the volume of the three-sphere of radius two.
- iii. **The ratio $2^3 = 8$ of Corollary 3.4:** a ratio of quantities of the same dimension, hence scale-invariant.

Corollary 3.4 (Volume Ratio of Cocoon to Patch)

From Theorem 3.3 and Corollary 3.2, the ratio of the $16\pi^2$ toroidal cocoon to the $2\pi^2$ mass patch is rigorously determined as the volume scaling accompanying a doubling of radius:

$$\frac{C_{\text{Torus}}}{m^{(4)}} = \frac{\text{Vol}(S_{R=2}^3)}{\text{Vol}(S_{R=1}^3)} = \frac{16\pi^2}{2\pi^2} = 2^3 = 8$$

3.2 Geometric Significance

Theorem 3.3 endows the quantity $16\pi^2 = (4\pi)^2$, recurring throughout this corpus, with a substance beyond algebraic multiplication. Before it is “the solid angle 4π multiplied by itself,” $16\pi^2$ is a **single geometric object: the three-sphere of radius two**, within which the Hopf fibre structure divides base and fibre into 4π each.

Taking $R = 2$ under the standard metric, both factors of the Hopf decomposition take the numerical value 4π , so that the decomposition $16\pi^2 = 4\pi \times 4\pi$ is realised as an **actual geometric decomposition**. This supplies a referent for the corpus’s reading of $16\pi^2$ as $4\pi \times 4\pi$ (the module rule of Paper 13). The order is “ R is chosen so that 4π appears”; that $16\pi^2$ is the volume of the three-sphere of radius two holds independently of how $R = 2$ is read.

4. Information-Geometric Interpretation of Exotic $\mathbb{R}^4$

Proposition 4.1 (Determination of the Volume Ratio)

The ratio between the three-sphere of radius one $S^3_{R=1}$ (volume $2\pi^2$) and that of radius two $S^3_{R=2}$ (volume $16\pi^2$) is fixed by Corollary 3.4 as $2^3 = 8$.

What this ratio compares is **two three-spheres**:

$$\frac{\text{Vol}(S^3_{R=2})}{\text{Vol}(S^3_{R=1})} = \frac{16\pi^2}{2\pi^2} = 8$$

Dividing $S^3_{R=2}$ (volume $16\pi^2$) by S^2 (area 4π) gives $16\pi^2/4\pi = 4\pi \approx 12.57$, but that is a ratio of a volume $[L^3]$ to an area $[L^2]$ and needs a further convention to carry meaning. The number this paper uses is the ratio **8** of quantities of the same dimension.

A Reading of the Topological Implications

What follows is a **reading** based on the established facts above.

The fact that uncountably many exotic $\mathbb{R}^4$ exist in four-dimensional Euclidean space alone may be read, from the standpoint of the Universe OS, as a phenomenon in which the freedom of topological connection is maximised in the data transformation from the 3D rendering patch 4π to the 4D protective cocoon $16\pi^2$. The picture is one in which, while topologically a single $\mathbb{R}^4$, the internal data-arrangement protocol (the differentiable structure) may branch.

Whether a rigorous correspondence exists between the realisability of intersection forms in Donaldson theory and the volume scaling of this paper is a **task for the future**. This section points to the possibility of a structural correspondence between the two.

Work in the opposite direction also exists on the physics side: Król and Asselmeyer-Maluga (2025), arguing from quantum gravity, suggest that exotic smooth four-spheres act as gravitational instantons and that the smooth four-dimensional Poincaré conjecture would thereby be negated. This paper does not take that route; it treats the smooth four-dimensional Poincaré conjecture as open and reads established results structurally.

5. Conclusion

This paper has shown the following.

1. By the $n = 3$ uniqueness theorem, the four-dimensional hyperspherical mass patch is uniquely fixed as $m^{(4)} = \text{Vol}(S^3) = 2\pi^2$ (Theorem 2.1).
2. By the Hopf decomposition, for every radius R the product of base area πR^2 and fibre circumference $2\pi R$ equals the total space volume $2\pi^2 R^3$ (Theorem 3.1).
3. At $R = 2$ both factors take the numerical value 4π , fixing the $16\pi^2$ toroidal cocoon as **a single geometric object: the three-sphere of radius two** (Theorem 3.3). The radius of agreement moves with the unit of length; what does not depend on units is the volume ratio $2^3 = 8$ (Corollary 3.4).
4. The relation to exotic $\mathbb{R}^4$ has been presented with established geometric fact and the reading built on it kept apart (Section 4).

Before it is 4π taken twice, $16\pi^2$ is the volume of the three-sphere of radius two. Inside it the Hopf fibre structure divides base and fibre into 4π each; the units decide at which radius that division shows, and only the factor **8** that comes with doubling the radius chooses no units.

Revision Record

4th edition (September 2026): The format was brought into line with the current template (a scope note directly after the abstract, the revision record moved from the front to just before the references, and the closing “statement of scope” gathered into the scope note). In the title, “Topological Elucidation” became “A Structural Interpretation.” In Section 1.2 the attribution of the uncountability of exotic $\mathbb{R}^4$ was corrected (the infinite family is due to Gompf, the uncountability to Taubes). In Section 4, Król and Asselmeyer-Maluga (2025) are cited as related work.

3rd edition (July 2026): The reading of $R = 2$ in Theorem 3.3 as the critical radius of geometric balance was removed, and the difference in dimension between an area and a length was written into Section 3. The objects of the ratio cited in Proposition 4.1 were corrected.

2nd edition (July 2026): The Hopf decomposition was recast in general form with the radius R as an explicit variable

(Theorem 3.1), and $16\pi^2$ was identified as the volume of the three-sphere of radius two (Theorem 3.3). In Section 4, established algebraic fact and topological reading were separated.

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