A Celestial Fix Improves with a Right-Angle Crossing Against Random Error and with Surrounding Stars Against Common Bias

— against random altitude error a 90 degree crossing does 5.758770 times better than 10 degrees, but a common bias shrinks as the crossing narrows, and three stars 120 degrees apart cancel it to 0 — the separator is whether the error is random or a bias common to all stars — [Paper 807]

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— From thinking to syncing. —

Abstract

Measuring a star's altitude places the observer on a circle centred beneath the star, which locally is a single line of position.When two or more lines of position are crossed to fix position, it is counted how stars should be chosen to make the position error small, separately for random altitude error and for a bias common to all stars.No new theorem or law is claimed. First, random error is smallest at a right-angle crossing——for two stars, 1.414214 arcminutes at 90 degrees and 8.144131 at 10 degrees, a ratio of 5.758770 (Section 2). **Second, and this is the core. A common bias is smaller the narrower the crossing——the shift from a 2-arcminute bias is 2.828427 arcminutes at 90 degrees and 2.007640 at 10 degrees, opposite in direction to random error** (Section 3). **Third, surrounding with stars makes a common bias vanish——three stars 120 degrees apart or four stars 90 degrees apart give a bias shift of 0** (Section 4). **Fourth, it vanishes only where the unit vectors of azimuth sum to 0——three stars at 0, 45 and 90 degrees have a sum of length 2.414214, and a bias shift of 2.414214 arcminutes remains** (Section 4). **Fifth, with both errors, the surrounding arrangement wins——the combined error is 3.162278 arcminutes for two stars at right angles and 1.154701 arcminutes (2.138505 km) for three stars 120 degrees apart** (Section 4). **Sixth, the separator is whether the error is random or a bias common to all stars——"choose stars that cross at right angles" is not decided true or false until one says which error one is up against** (Section 5). **Keywords**: celestial fix, line of position, crossing angle, common bias, least squares, arrangement of stars.

Scope of this paper (scope note): **No new theorem or law is claimed——drawing lines of position from star altitudes, crossing them by least squares, the crossing angle setting the position error, and surrounding stars cancelling a common error are all known.A plane approximation——lines of position are straight, and the Earth's curvature is left out.Only two kinds of error——random altitude error (independent normal for each star) and a bias of the same size on all stars.The numbers are examples——a random error of 1 arcminute, a bias of 2 arcminutes and 20000 random draws are values for comparison.The bias is not solved as an unknown——only the two position coordinates are found by least squares; solving the bias as a third unknown at the same time is not entered.Relation to earlier papers**: **Paper 334 showed that what is being added is not the values but the precisions, and its scope note stated "bias is not treated —each sensor is taken as unbiased; combining systematic errors is not the subject"——this paper sits in that seat, counting how a bias common to all stars, when lines of position are combined, reaches the position depending on the arrangement.Paper 255 showed that "accuracy" is two things and calibration removes only one——there calibration removed the bias. Here the arrangement of stars removes it without calibration.Paper 340 showed that the shorter path is not the one a navigator steers, and its scope note stated that navigation was not taught——this paper stands beside that seat, teaching neither the actual choice of stars nor the correction procedures for dip, refraction and semidiameter.**What is added is **showing by formula and random numbers that random error and common bias respond to the crossing angle in opposite directions**, **confirming that the arrangements where bias vanishes coincide with those where the unit vectors of azimuth sum to 0**, **setting the combined error side by side for six arrangements**, and **placing the separator on random error versus common bias**.

1. Notation and the Question

Setting: **misreading the altitude of a star at azimuth A_j by δh_j shifts its line of position toward the star by δh_j. The position shift (x, y) is found by least squares.**

$$x\cos A_j + y\sin A_j = \delta h_j, \qquad \begin{pmatrix} x \\ y \end{pmatrix} = (G^{\mathsf{T}}G)^{-1}G^{\mathsf{T}}\,\delta h, \qquad G = \begin{pmatrix} \cos A_1 & \sin A_1 \\ \vdots & \vdots \end{pmatrix}$$

For random error $\delta h_j \sim \mathcal{N}(0, \sigma^2)$ the root-mean-square position error is $\sigma\sqrt{\operatorname{tr}(G^{\mathsf{T}}G)^{-1}}$; for a common bias $\delta h_j = b$ the position shift is $b(G^{\mathsf{T}}G)^{-1}G^{\mathsf{T}}\mathbf{1}$.

Question: **does the choice of stars that makes the position error small change with the kind of error?**

2. Random Error

Computation 2.1: **with random error $\sigma = 1$ arcminute, the root-mean-square position error for each arrangement was derived by formula and by random numbers (20000 draws).**

arrangement	formula (arcmin)	random (arcmin)
two stars, crossing 90 deg	1.414214	1.407629
two stars, crossing 30 deg	2.828427	2.840193
two stars, crossing 10 deg	8.144131	8.125672
three stars, 0/45/90 deg	1.224745	1.222437
three stars, 120 deg apart	1.154701	1.161214
four stars, 90 deg apart	1.000000	1.001635

> **Observation 2.2**
>
> **random error is smallest at a right-angle crossing.For two stars, by the following formula, a 10-degree crossing is 5.758770 times a 90-degree one ($1/\sin 10°$)——two lines crossing shallowly let a small shift slide the crossing point far.**

$$\text{two stars:}\quad \sqrt{2}\,\sigma/\sin\theta$$

3. Common Bias

Computation 3.1: **the same bias $b = 2$ arcminutes was put on all stars and the position shift derived.**

crossing angle	random-error formula $\sqrt{2}\sigma/\sin\theta$	bias-shift formula $b/\cos(\theta/2)$
90 deg	1.414214	2.828427
60 deg	1.632993	2.309401
30 deg	2.828427	2.070552
10 deg	8.144131	2.007640

Observation 3.2

this is the core: a common bias is smaller the narrower the crossing.The bias shift is 2.828427 arcminutes (5.238247 km) at 90 degrees and 2.007640 arcminutes at 10 degrees, a 10-to-90 ratio of 0.709808, opposite in direction to the random-error ratio of 5.758770——when two lines shift the same way, the crossing point moves along their bisector, and farther the wider the lines open.

4. Surrounding Arrangements

Computation 4.1: **for arrangements of three and four stars, the bias shift and the length of the sum of azimuth unit vectors were set side by side, and the combined error $\sqrt{(\text{random})^2 + (\text{bias})^2}$ derived.**

arrangement	bias shift (arcmin)	length of unit-vector sum	combined error (arcmin, km)
two stars, crossing 90 deg	2.828427	1.414214	3.162278, 5.856538
two stars, crossing 30 deg	2.070552	1.931852	3.505308, 6.491831
two stars, crossing 10 deg	2.007640	1.992389	8.387937, 15.534460
three stars, 0/45/90 deg	2.414214	2.414214	2.707107, 5.013562
three stars, 120 deg apart	0.000000	0.000000	1.154701, 2.138505
four stars, 90 deg apart	0.000000	0.000000	1.000000, 1.852000

Observation 4.2

surrounding with stars makes a common bias vanish.The bias shift is $(G^{\mathsf{T}}G)^{-1}$ times the sum of azimuth unit vectors $G^{\mathsf{T}}\mathbf{1}$, so it is 0 only where that sum is 0 (three stars 120 degrees apart, four stars 90 degrees apart). Across the six arrangements, those with zero bias shift coincided with those whose sum has length 0 ——a bias equal on all stars cancels between stars in opposite directions.

Observation 4.3

with both errors, the surrounding arrangement wins.With a bias, three stars 120 degrees apart give 1.154701 arcminutes, smaller than the 3.162278 of two stars at right angles, while two stars crossing at 10 degrees give 8.387937——the right angle is best only against random error; against a bias, the surrounding arrangement is best.

5. The Separator

Observation 5.1

the separator is whether the error is random or a bias common to all stars.Random error is smallest at 90 degrees and 5.758770 times larger at 10; a common bias is smaller the narrower, 0.709808 times at 10 degrees, and 0 when surrounded——"choose stars that cross at right angles" is not decided true or false until one says which error one is up against.

Observation 5.2

beside Papers 334, 255 and 340.Paper 334 showed that precisions add for unbiased sensors and left bias aside, 255 that bias is removed by calibration, and 340 taught no navigation——here, sitting in the seat for bias that 334 left open, it was counted that arrangement removes the bias in place of 255's calibration.

Observation 5.3

what is not said.The Earth's curvature, the actual choice of stars, errors of time, altitude correction procedures and solving the bias as an unknown are not treated——what is shown is that the same arrangement acts in opposite directions on two kinds of error.

6. Roll Call and Closing

1. **random error at 90 deg, $\mathbf{1.414214}$ arcmin** (Computation 2.1): **right angle is smallest.**
2. **$\mathbf{8.144131}$ arcmin at 10 deg** (Computation 2.1): **a shallow crossing.**
3. **ratio $\mathbf{5.758770}$** (Observation 2.2): $\mathbf{1/\sin 10°}$.

4. **bias shift 2.828427 arcmin at 90 deg** (Computation 3.1): **farther the wider.**
5. **2.007640 arcmin at 10 deg** (Observation 3.2): **the core; opposite direction.**
6. **ratio 0.709808** (Observation 3.2): **smaller the narrower.**
7. **bias 0 at 120 deg apart** (Observation 4.2): **vanishes when surrounded.**
8. **2.414214 arcmin at 0/45/90 deg** (Computation 4.1): **remains unless the sum is 0.**
9. **combined 1.154701 arcmin (2.138505 km)** (Observation 4.3): **smaller than 3.162278 at right angles.**
10. **random error or common bias** (Observation 5.1): **the separator; Papers 334, 255, 340.**

Closing: **A celestial fix improves with a right-angle crossing against random error and with surrounding stars against a common bias, and the crossing angle acts in opposite directions on the two.A random altitude error of 1 arcminute gives 1.414214 arcminutes for two stars crossing at 90 degrees and 8.144131 at 10 degrees (5.758770 times)——shallowly crossing lines let the crossing slide.A 2-arcminute bias common to all stars does the opposite, 2.828427 arcminutes at 90 degrees and 2.007640 at 10, smaller the narrower, and 0 where the azimuth unit vectors sum to 0, so that with both errors three stars 120 degrees apart give 1.154701 arcminutes, smaller than the 3.162278 of two stars at right angles——a bias in the same direction cancels between stars in opposite directions.The separator is whether the error is random or a bias common to all stars.Placed among the earlier papers——sitting in the seat for bias left open by Paper 334, it was counted that arrangement removes the bias in place of Paper 255's calibration, standing beside Paper 340's seat for navigation.To be honest——a plane approximation was used, and neither the actual choice of stars, correction procedures, nor solving the bias as an unknown is treated.**

References

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