The Period of a Continued Fraction Marks Being a Root of a Quadratic, Not Being Easy to Approximate

— All **43** non-square $\sqrt{n}$ with n from **2** to **50** have a period (length from **1** to **12**, mean **3.441860**), and the golden ratio stays **1** for all **100** terms — the separator is whether there is a period or a one-off large partial quotient — [Paper 811]

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— From thinking to syncing. —

Abstract

Some continued fractions have a period and some do not, and sharp approximations such as $355/113$ for π also come out of a continued fraction.This paper expands them and counts what a period marks and where a sharp approximation comes from.No new theorem or law is claimed. First, square roots have a period without exception——all 43 non-square $\sqrt{n}$ for n from 2 to 50 have one, of length 1 to 12 with mean 3.441860 (Section 2). **Second, and this is the core. A period marks being a root of a quadratic——rebuilding from the periods $[2]$, $[1,2]$, $[1,1,1,4]$ returns $\sqrt{2}$, $\sqrt{3}$, $\sqrt{7}$, with residuals in x^2-n at the 10^{-16} level** (Section 3). **Third, some numbers have no period——the continued fraction of e breaks every period of length 1 to 30 within 100 terms, and its largest partial quotient there is 66** (Section 4). **Fourth, a sharp approximation comes from a one-off jump, not from a period——the error of $355/113$, 2.667642×10^{-7}, is 2.109660×10^{-4} times that of $22/7$, 1.264489×10^{-3}, because the partial quotient just before it is 292** (Section 4). **Fifth, the golden ratio sits at the opposite end——its continued fraction is 1 for all 100 terms, so no jump ever comes** (Section 2). **Sixth, the separator is whether there is a period or a one-off large partial quotient——"this continued fraction is beautiful" means different things under the two** (Section 5). **Keywords**: continued fraction, period, quadratic irrational, partial quotient, convergent, Lagrange's theorem.

Scope of this paper (scope note): **No new theorem or law is claimed——continued fraction expansions, the periodicity of quadratic irrationals (Lagrange), the shape of the expansions of π and e, and the sharpness of $355/113$ are all known.No theorem is proved——Lagrange's theorem is only cited, and the check stops at rebuilding the quadratic irrational from its period.Aperiodicity is not proved——for e all that is stated is the fact that no period of length up to 30 holds within 100 terms.Transcendence is not entered——that π and e are transcendental is not used.How easy a number is to approximate is not treated——which number is hardest to approximate, and where the lower bound of $q|q\alpha - p|$ settles, are not entered.Relation to earlier papers**: **Paper 319 showed that a calendar is three approximations stacked, and its scope note cited approximation by continued fractions as known——this paper stands beside that seat: there the stacked approximations of a calendar, here the single question of what a period marks.Paper 539 showed that what fixed the tooth counts was a continued fraction, not astronomy——there the continued fraction was a tool; here its own shape is examined.Paper 213 showed that "coming back" has distinct roots——there the return of an orbit, here the period of an expansion: the same word on a different stage.**What is added is **measuring the period lengths of 43 square roots for a mean of 3.441860, rebuilding numbers from periods and measuring the residuals, measuring the sharpness of $355/113$ as 2.109660×10^{-4} times that of $22/7$ and tracing it to the one-off 292**, and **placing the separator on period versus one-off jump**.

1. Notation and the Question

Setting: **write the continued fraction of a real x as $x = a_0 + \dfrac{1}{a_1 + \dfrac{1}{a_2 + \cdots}}$ and call $a_0, a_1, a_2, \ldots$ the partial quotients.The fraction p/q obtained by truncating is a convergent.**

In double precision the expansion breaks after about 40 terms, so the computation used 120 significant digits.That the golden ratio keeps partial quotient 1 for 100 terms was used as a check on the precision itself.

Question: **what does a period mark, and where does a sharp approximation come from?**

2. Numbers With a Period

Computation 2.1: **the continued fractions of $\sqrt{n}$ were expanded in integer arithmetic and their period lengths measured.**

number	start of the expansion	period length
golden ratio	$[1, 1, 1, 1, 1, \ldots]$	1
$\sqrt{2}$	$[1, 2, 2, 2, \ldots]$	1
$\sqrt{3}$	$[1, 1, 2, 1, 2, \ldots]$	2
$\sqrt{7}$	$[2, 1, 1, 1, 4, \ldots]$	4
$\sqrt{13}$	$[3, 1, 1, 1, 1, 6, \ldots]$	5
$\sqrt{19}$	$[4, 2, 1, 3, 1, 2, 8, \ldots]$	6

Observation 2.2

square roots have a period without exception.All 43 non-square values from 2 to 50 have one, of length 1 to 12 with mean 3.441860——the golden ratio, with period 1 and every partial quotient equal to 1, sits at the end where no jump ever comes.

3. Rebuilding a Number From Its Period

Computation 3.1: **given only the period, the fixed point of $x = a_0 + 1/y$ with y the infinitely repeated period was solved numerically and compared with the original number.**

period given	number rebuilt	residual in $x^2 - n$
$[2]$	1.414213562373	4.441×10^{-16}
$[1, 2]$	1.732050807569	4.441×10^{-16}
$[1, 1, 1, 4]$	2.645751311065	8.882×10^{-16}

Observation 3.2

this is the core. A period marks being a root of a quadratic.An expression that repeats its period returns to itself, so its fixed point is an equation between fractions, and clearing denominators gives a quadratic—— this is what Lagrange's theorem says: having a period and being a quadratic irrational are two ways of stating one fact.

4. Numbers Without a Period, and the One-Off Jump

Computation 4.1: **the continued fractions of π and e were expanded.**

number	continued fraction
π	$[3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14]$
e	$[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots]$

Within 100 terms e satisfies no period of length 1 to 30, and its largest partial quotient there is 66.For $\sqrt{2}$ the largest is 2, and for the golden ratio 1.

Computation 4.2: **the errors of the convergents of π were listed.**

convergent	value	error against π
$22/7$	3.142857142857	1.264489×10^{-3}
$333/106$	3.141509433962	8.321963×10^{-5}
$355/113$	3.141592920354	2.667642×10^{-7}
$103993/33102$	3.141592653012	5.778906×10^{-10}

Observation 4.3

a sharp approximation comes from a one-off jump, not from a period.The error of $355/113$ is 2.109660×10^{-4} times that of $22/7$, and the partial quotient just before it is 292——the larger the next partial quotient, the sharper the convergent truncated before it; the 292 of π occurs once and is not a period.

5. The Separator

Observation 5.1

the separator is whether there is a period or a one-off large partial quotient.$\sqrt{7}$ has period 4 but no partial quotient beyond 4, while π has no period yet shows a 292 once——"this continued fraction is beautiful" says one thing if it means regularity and another if it means a sharp approximation.

Observation 5.2

placed beside Papers 319, 539 and 213.Paper 319 read a calendar as three stacked approximations, Paper 539 showed tooth counts fixed by a continued fraction, and Paper 213 separated the roots of "coming back"——here the period of the expansion itself is examined and found to mark a root of a quadratic.

Observation 5.3

what is not said.The proof of Lagrange's theorem, a proof that e has no period, transcendence, and how easy a number is to approximate are not treated——what is shown is what a period marks and where a sharp approximation comes from.

6. Census and Closing

1. **all 43 square roots have a period** (Observation 2.2): **no exception.**
2. **mean period length 3.441860** (Observation 2.2): **from 1 to 12.**
3. **golden ratio all 1 for 100 terms** (Computation 2.1): **the end with no jump.**
4. **period $[2]$ gives 1.414213562373** (Computation 3.1): **back to $\sqrt{2}$.**
5. **residuals at 10^{-16}** (Computation 3.1): **the core; a root of a quadratic.**
6. **e breaks every period to length 30** (Computation 4.1): **largest partial quotient 66.**
7. **error of $22/7$: 1.264489×10^{-3}** (Computation 4.2): **the first hit.**
8. **error of $355/113$: 2.667642×10^{-7}** (Computation 4.2): **a sharp one.**
9. **ratio 2.109660×10^{-4}** (Observation 4.3): **it comes from 292.**
10. **period or one-off jump** (Observation 5.1): **the separator; Papers 319, 539, 213.**

Closing: **The period of a continued fraction marks being a root of a quadratic, not being easy to approximate.All 43 non-square $\sqrt{n}$ from 2 to 50 have a period (mean length 3.441860), and rebuilding from the period alone returns $\sqrt{2}$, $\sqrt{3}$, $\sqrt{7}$ with residuals in $x^2 - n$ at the 10^{-16} level——because an expression repeating its period returns to itself, and its fixed point is a quadratic.Neither π nor e has a period (e breaks every period of length up to 30 within 100 terms), and the sharp $355/113$ (error 2.667642×10^{-7}, 2.109660×10^{-4} times that of $22/7$) comes from the one-off partial quotient 292——the larger the next partial quotient, the sharper the convergent truncated before it.The separator is whether there is a period or a one-off large partial quotient.Placed among the earlier papers——standing beside Paper 319's seat for continued fractions, examining the shape of the tool Paper 539 used, and keeping a different stage from Paper 213's "coming back".To be honest——Lagrange's theorem is cited and not proved, and the aperiodicity of e is a fact within 100 terms, not a proof.**

References

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