What Decides How Easily a Number Is Approximated Is Whether Its Partial Quotients Are Bounded

— For convergents p/q, the quantity $q|q\alpha - p|$ settles at 0.447214 $(= 1/\sqrt{5})$ for the golden ratio, 0.353553 for $\sqrt{2}$, 0.288675 for $\sqrt{3}$ and 0.223607 for $\sqrt{5}$ — the separator is whether the partial quotients are bounded — [Paper 812]

Yuuki YAMAGISHI (山岸 勇輝)
Yamagishi Complexity Torus Lab, Japan
September 2026

— From thinking to syncing. —

Abstract

When one says "this number is easy to approximate", what is actually doing the work is counted here.For each convergent p/q the quantity $q|q\alpha - p|$ was computed, and where it settles was compared across six numbers.No new theorem or law is claimed. First, for numbers with bounded partial quotients it settles at a positive value——0.447214 for the golden ratio, 0.353553 for $\sqrt{2}$, 0.288675 for $\sqrt{3}$ and 0.223607 for $\sqrt{5}$ (Section 2). **Second, those values match closed forms——the differences against $1/\sqrt{5} = 0.447214$, $1/(2\sqrt{2}) = 0.353553$ and $1/\sqrt{12} = 0.288675$ are all at the 10^{-16} level** (Section 2). **Third, and this is the core. For numbers with unbounded partial quotients it keeps falling——π reaches 0.003406, and e falls from 0.037012 to 0.024383 merely on extending from 40 to 60 terms** (Section 3). **Fourth, the divide is not what the numbers are but how large the partial quotients grow——the largest is 1 for the golden ratio, 2 for $\sqrt{2}$, 2 for $\sqrt{3}$ and 4 for $\sqrt{5}$, against 292 for π and 40 for e** (Section 3). **Fifth, the golden ratio sits at the end——its value 0.447214 coincides with Hurwitz's constant $1/\sqrt{5}$ and is 18.341223 times the 0.024383 of e** (Section 4). **Sixth, the separator is whether the partial quotients are bounded——"all irrationals are about equally hard to approximate" does not hold, and what divides them is the shape of the expansion, not the name of the number** (Section 4). **Keywords**: Diophantine approximation, continued fraction, partial quotient, Hurwitz's theorem, golden ratio, convergent.

Scope of this paper (scope note): **No new theorem or law is claimed——Diophantine approximation, the best-approximation property of convergents, Hurwitz's theorem with $1/\sqrt{5}$ as the best constant, and the shapes of the expansions of π and e are all known.No theorem is proved——Hurwitz's theorem is only cited, and the check stops at the golden ratio's value coinciding with $1/\sqrt{5}$.Unboundedness is not proved——the 292 of π and the growing partial quotients of e are facts seen within 25 and 60 terms.Irrationality measure is not entered——Liouville numbers, transcendence and measures of irrationality are not the subject.Only convergents are examined——not the best approximation over all q, but the convergents of the continued fraction.Relation to earlier papers**: **Paper 811 showed that the period of a continued fraction marks being a root of a quadratic——this paper stands beside it: numbers with a period have bounded partial quotients and so fall on the side where the bound settles at a positive value.Paper 176 showed that "small" is two words that do not agree——this paper likewise separates two readings of "easy to approximate".Paper 300 showed that whether two things share a root is decidable——values such as 0.447214 occur elsewhere in the corpus as universal constants, with different content, and are not written up as sharing a root.**What is added is **measuring $q|q\alpha - p|$ for six numbers in high precision and confirming agreement with closed forms at the 10^{-16} level**, **setting the largest partial quotient beside the way the bound settles**, **measuring the golden ratio's value as 18.341223 times that of e**, and **placing the separator on bounded versus unbounded partial quotients**.

1. Notation and the Question

Setting: **write p/q for a convergent of the continued fraction of an irrational α, and measure the quality of the approximation by $q|q\alpha - p|$.The smaller this is, the better the fraction does for its denominator.**

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{c\,q^2} \quad \Longleftrightarrow \quad q\,|q\alpha - p| < \frac{1}{c}$$

In double precision the expansion breaks after about 40 terms, so the computation used 120 significant digits.

Question: **where does this quantity settle, and what decides how it settles?**

2. Numbers With Bounded Partial Quotients

Computation 2.1: **convergents were built for 40 terms and the settled value of $q|q\alpha - p|$ compared with closed forms.**

number	settled value	closed form	difference
golden ratio	0.447214	$1/\sqrt{5} = 0.447214$	0
$\sqrt{2}$	0.353553	$1/(2\sqrt{2}) = 0.353553$	5.551×10^{-17}
$\sqrt{3}$	0.288675	$1/\sqrt{12} = 0.288675$	5.551×10^{-17}
$\sqrt{5}$	0.223607	—	—

Observation 2.2

for numbers with bounded partial quotients it settles at a positive value.None of them falls away; each hovers about a fixed value, agreeing with the closed form at the 10^{-16} level——all of these have periodic continued fractions, with no partial quotient above 4.

3. Numbers With Unbounded Partial Quotients

Computation 3.1: **the same quantity was measured for π and e and set beside the largest partial quotient.**

number	smallest $q\lvert q\alpha - p\rvert$	largest partial quotient
golden ratio (40 terms)	0.447214	1
$\sqrt{2}$ (40 terms)	0.353553	2
$\sqrt{5}$ (40 terms)	0.223607	4
π (25 terms)	0.003406	292
e (40 terms)	0.037012	26
e (60 terms)	0.024383	40

Observation 3.2

this is the core. For numbers with unbounded partial quotients it keeps falling.Merely extending e from 40 to 60 terms drops the smallest value from 0.037012 to 0.024383, and a single 292 takes π down to 0.003406 ——a large partial quotient makes the convergent truncated just before it a sharp one, so while the partial quotients keep growing the bound never settles.

Observation 3.3

the divide is not what the numbers are but the shape of the expansion.$\sqrt{2}$ and π are both irrational, yet their largest partial quotients are 2 and 292, and they settle differently——"all irrationals are about equally hard to approximate" does not hold.

4. The Number at the End, and the Separator

Observation 4.1

the golden ratio sits at the end.Its value 0.447214 coincides with Hurwitz's constant $1/\sqrt{5}$ and is 18.341223 times the 0.024383 of e over 60 terms——having every partial quotient equal to 1 means no jump ever comes, and that is the hardest shape to approximate.

Observation 4.2

the separator is whether the partial quotients are bounded.Numbers with a period (the quadratic irrationals) have bounded partial quotients and settle at a positive value, while π and e keep falling——the phrase "easy to approximate" attaches to the shape of the expansion, not to the name of the number.

Observation 4.3

placed beside Papers 811, 176 and 300.Paper 811 showed that a period marks being a root of a quadratic——such numbers fall on the bounded side here, so the two results are two faces of one boundary.Paper 176 showed that "small" is two words that do not agree, and Paper 300 that sharing a root is decidable——values such as 0.447214 appear elsewhere as universal constants with different content.

Observation 4.4

what is not said.The proof of Hurwitz's theorem, a proof that the partial quotients of π and e are unbounded, irrationality measures and transcendence are not treated——what is shown is that the boundedness of the partial quotients decides how the quantity settles.

5. Census and Closing

1. **golden ratio 0.447214** (Computation 2.1): **difference 0 from $1/\sqrt{5}$.**
2. **$\sqrt{2}$ 0.353553** (Computation 2.1): **$1/(2\sqrt{2})$.**
3. **$\sqrt{3}$ 0.288675** (Computation 2.1): **$1/\sqrt{12}$.**
4. **$\sqrt{5}$ 0.223607** (Computation 2.1): **the bounded side.**
5. **π 0.003406** (Computation 3.1): **partial quotient 292.**
6. **e from 0.037012 to 0.024383** (Observation 3.2): **the core; it keeps falling.**
7. **largest partial quotients 1, 2, 4** (Computation 3.1): **the bounded side.**
8. **largest partial quotients 292, 40** (Computation 3.1): **the unbounded side.**
9. **golden ratio is 18.341223 times e** (Observation 4.1): **hardest to approximate.**
10. **bounded or unbounded** (Observation 4.2): **the separator; Papers 811, 176, 300.**

Closing: **What decides how easily a number is approximated is whether its partial quotients are bounded.The quantity $q|q\alpha - p|$ settles at 0.447214 for the golden ratio, 0.353553 for $\sqrt{2}$, 0.288675 for $\sqrt{3}$ and 0.223607 for $\sqrt{5}$, agreeing with closed forms at the 10^{-16} level——none of these has a partial quotient above 4.For π it reaches 0.003406, and for e extending from 40 to 60 terms alone drops it from 0.037012 to 0.024383——a large partial quotient sharpens the convergent just before it, so while they keep growing the bound never settles.The golden ratio's 0.447214 coincides with Hurwitz's constant $1/\sqrt{5}$ and is 18.341223 times the value for e, placing it at the hardest end.The separator is whether the partial quotients are bounded.Placed among the earlier papers——**

standing beside Paper 811's result on periods (those numbers fall on the bounded side here), beside Paper 176's distinct roots of "small", and keeping Paper 300's discipline about universal constants.To be honest——Hurwitz's theorem is cited and not proved, and the unboundedness for π and e is a fact within 25 and 60 terms.

References

1. Yamagishi, Y. (2026). *"Small" Is Two Words, and They Do Not Agree* [Paper 176]. Zenodo. 10.5281/zenodo.22097994
2. Yamagishi, Y. (2026). *The Three Hundredth Census: Whether Two Things Share a Root Is Decidable* [Paper 300]. Zenodo. 10.5281/zenodo.22171914
3. Yamagishi, Y. (2026). *The Period of a Continued Fraction Marks Being a Root of a Quadratic, Not Being Easy to Approximate* [Paper 811]. Zenodo. 10.5281/zenodo.23015470

— *From thinking to syncing.* —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)