The Optimality of Chebyshev Nodes Is Not a Comparison Won but an Equality

— the maximum of the node polynomial on $[-1, 1]$ equals 2^{1-n} at ratio 1.000000 for Chebyshev nodes, and 300 random node sets never fell below it, the smallest being 3.559066 times — the separator is whether "best" means a comparison won or an equality with a floor — [Paper 818]

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— *From thinking to syncing.* —

Abstract

Where the nodes of an interpolation are placed changes how the error behaves.This paper checks in numbers that the merit of Chebyshev nodes is not "it worked well when tried" but "it cannot be done better".No new theorem or law is claimed. First, with Chebyshev nodes the maximum lands exactly on a closed form——the ratio to 2^{1-n} is 1.000000 at $n = 5, 9, 13, 17, 21$ (Section 2). **Second, that value falls exponentially with n——2.441406×10^{-4} at $n = 13$ and 9.536743×10^{-7} at $n = 21$** (Section 2). **Third, equispaced nodes give more, and the gap widens with n ——1.815716 times at $n = 5$, 16.416632 at $n = 13$, 245.018559 at $n = 21$** (Section 3). **Fourth, and this is the core. Random nodes never break the bound——over 300 draws at $n = 9$ the smallest was 3.559066 times the bound, with 0 falling below** (Section 3). **Fifth, so "Chebyshev nodes are best" is not the outcome of a comparison—— 2^{1-n} is a floor no one can pass, and Chebyshev nodes touch it** (Section 4). **Sixth, the separator is whether "best" means a comparison won or an equality with a floor——the word carries different content in the two cases** (Section 4). **Keywords**: Chebyshev nodes, node polynomial, minimax, lower bound, equispaced nodes, interpolation.

Scope of this paper (scope note): **No new theorem or law is claimed——Chebyshev polynomials, that the maximum of a monic polynomial is minimised at 2^{1-n}, and that Chebyshev nodes attain it, are all known.No theorem is proved——the proof of optimality is not entered; only the agreement of the equality and the failure of random nodes to break the floor are checked.The interval is fixed——only $[-1, 1]$ is examined; mapping to other intervals is not treated.Interpolation error is not entered——Runge's phenomenon and conditions for convergence are not the subject.Weights are not treated——weighted best approximation, Chebyshev points of the second kind and node sets including the endpoints are not treated.Relation to earlier papers**: **Paper 329 showed that on the same points one method diverges by a thousand while another converges by a thousandth, and its scope note cited Runge's 1901 example, Chebyshev points, cubic splines and the Lebesgue constant as standard——this paper stands beside that seat: there the points were fixed and the methods compared, here the method is fixed and the points examined.Paper 300 showed that whether two things share a root is decidable——"better when compared" and "cannot be bettered" are distinct roots and are not put under one word.**What is added is **measuring the maximum at five values of n and confirming the ratio 1.000000 to 2^{1-n}, measuring the gap to equispaced nodes widening from 1.815716 to 245.018559, confirming that 300 random node sets never break the bound**, and **placing the separator on a comparison versus an equality**.

1. Notation and the Question

Setting: **place n nodes $x_1, \ldots, x_n$ in $[-1, 1]$ and consider**

$$w(x) = \prod_{i=1}^{n} (x - x_i), \qquad M(\{x_i\}) = \max_{x \in [-1,1]} |w(x)|$$

This w is monic and appears inside the interpolation error.The question is how small M can be made by choosing the nodes.

Question: **does M have a floor, and if so do Chebyshev nodes touch it?**

2. Measuring at Chebyshev Nodes

Computation 2.1: **with $x_k = \cos \frac{(2k+1)\pi}{2n}$, M was measured and compared with 2^{1-n}.**

n	measured M	2^{1-n}	ratio
5	6.250000×10^{-2}	6.250000×10^{-2}	1.000000
13	2.441406×10^{-4}	2.441406×10^{-4}	1.000000
21	9.536743×10^{-7}	9.536743×10^{-7}	1.000000

> **Observation 2.2**
>
> it lands exactly.The ratio is 1.000000 at all five values of n——at Chebyshev nodes w oscillates with peaks and troughs of equal height $n+1$ times across the interval, and that height is 2^{1-n}.

3. Comparing Other Nodes

Computation 3.1: the same quantity was measured at equispaced nodes.

n	equispaced M	times the Chebyshev value
5	1.134822×10^{-1}	1.815716
13	4.007967×10^{-3}	16.416632
21	2.336679×10^{-4}	245.018559

Computation 3.2: at $n = 9$, nodes were drawn at random from the interval 300 times.

measured	value
bound 2^{1-9}	3.906250×10^{-3}
smallest M among 300 draws	1.390260×10^{-2}
ratio to the bound	3.559066
draws falling below the bound	0

Observation 3.3

this is the core. Random nodes never break the bound.The smallest of 300 draws is still 3.559066 times the bound, and none fell below——luck can find a good set, but it cannot pass through the floor.

4. The Separator

Observation 4.1

the separator is whether “best” means a comparison won or an equality with a floor.The factors against equispaced nodes (1.815716 to 245.018559) come from a comparison, while 2^{1-n} has no counterpart to compare with——“Chebyshev nodes are best” does not mean best among those tried; it means touching the floor.

Observation 4.2

a floor existing and a choice touching it are two things.A bound may exist with nothing attaining it——here both hold, and the ratio came out at 1.000000.

Observation 4.3

placed beside Papers 329 and 300.Paper 329 showed one method diverging by a thousand and another converging by a thousandth on the same points——there the points were fixed and the methods differed; here the method is fixed and the points differ.Paper 300 showed that sharing a root is decidable——“better when compared” and “cannot be bettered” are distinct roots.

Observation 4.4

what is not said.The proof of optimality, Runge’s phenomenon, the Lebesgue constant and weighted approximation are not treated——what is shown is that the merit here is an equality.

5. Census and Closing

1. $\mathbf{2.441406 \times 10^{-4}}$ **at** $\mathbf{n = 13}$ (Computation 2.1): **equal to** $\mathbf{2^{1-n}}$**.**
2. $\mathbf{9.536743 \times 10^{-7}}$ **at** $\mathbf{n = 21}$ (Computation 2.1): **falling exponentially.**
3. **all ratios** $\mathbf{1.000000}$ (Observation 2.2): **landing exactly.**
4. **equispaced** $\mathbf{1.815716}$ **times at** $\mathbf{n = 5}$ (Computation 3.1): **close at small** $\mathbf{n}$**.**
5. $\mathbf{16.416632}$ **times at** $\mathbf{n = 13}$ (Computation 3.1): **the gap opens.**
6. $\mathbf{245.018559}$ **times at** $\mathbf{n = 21}$ (Computation 3.1): **widening with** $\mathbf{n}$**.**
7. **smallest random** 1.390260×10^{-2} (Computation 3.2): $\mathbf{3.559066}$ **times the bound.**
8. $\mathbf{0}$ **draws below the bound** (Observation 3.3): **the core; the floor holds.**
9. **floor and attainment** (Observation 4.2): **both present here.**
10. **comparison or equality** (Observation 4.1): **the separator; Papers 329, 300.**

Closing: **The optimality of Chebyshev nodes is not a comparison won but an equality.The maximum of the node polynomial lands exactly on** $\mathbf{2^{1-n}}$ **at Chebyshev nodes (ratio** $\mathbf{1.000000}$ **at five values of** $\mathbf{n}$**;** $\mathbf{2.441406 \times 10^{-4}}$ **at** $\mathbf{n = 13}$ **and** $\mathbf{9.536743 \times 10^{-7}}$ **at** $\mathbf{n = 21}$**)——because the peaks and troughs come out equal in height.Equispaced nodes give more, from** $\mathbf{1.815716}$ **times at** $\mathbf{n = 5}$ **to** $\mathbf{245.018559}$ **at** $\mathbf{n = 21}$**, and of** $\mathbf{300}$ **random node sets the smallest was still** $\mathbf{3.559066}$ **times the bound with none below it——luck can find a good set but cannot pass the floor.The separator is whether "best" means a comparison won or an equality with a floor.Placed among the earlier papers——standing beside Paper 329's seat (there the points fixed and the methods compared), and keeping Paper 300's discipline between "better" and "cannot be bettered".To be honest——optimality itself is not proved; what was checked is the equality and the failure of random nodes to break the floor.**

References

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