

The Procedure Is Not Determined, but Its Parity Is

— Counting the ways to build the four-cycle $(1, 2, 3, 0)$ as a product of swaps gives **16** ways of length **3** and **640** of length **5** (forty times as many), while no even length works at all — the separator is whether one asks about the length of the procedure or about its parity — [Paper 841]

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— *From thinking to syncing.* —

Abstract

Two people rearranging the same things take different steps. This paper counts the one thing that does not change when they do. No new theorem or law is claimed. First, the procedure is not unique: the four-cycle has **16** decompositions of length **3** and **640** of length **5** (Section 2). Second, and this is the core. Not even the length is unique: both **3** and **5** work, and the longer one is forty times as common (Section 2). Third, and yet the parity is: no even length works at all (0 ways) (Section 2). Fourth, that parity agrees with two other counts: across all **24** permutations of $n = 4$, the inversion count and n minus the number of cycles never disagreed in parity (0 cases) (Section 3). Fifth, even permutations are exactly half: **12** of the **24**, a share of **0.500000** (Section 3). Sixth, this uniqueness is what lets a determinant be written: the signed sum **49.500000000** and elimination **49.500000000** agree to 7.105×10^{-15} (Section 4). **Keywords:** sign of a permutation, transposition, inversion count, determinant, exhaustive enumeration, parity.

Scope of this paper (scope note): No new theorem or law is claimed: that the sign of a permutation does not depend on the decomposition, that the parity of the inversion count matches that of n minus the number of cycles, that the alternating group has order $n!/2$, and that the determinant has the Leibniz form, are all known. No group theory is built: simplicity of the alternating group and representation theory are not entered. Real rearrangements are not discussed: sorting costs and the ordering of actual work are not the subject. Puzzles are not treated: moving pieces on a board belongs to Paper 601. The enumeration is not widened: decompositions of length 7 and beyond were not counted. Relation to earlier papers: Paper 601 showed that half the positions of the 15 puzzle are unreachable, and its scope note said that its parity is the parity of the arrangement (the sign of a permutation matrix), using the sign rather than examining it. This paper does not sit in that seat but stands beside it: there the shape of the board decides whether parity bites, here why parity is single-valued at all. Paper 247 counted eight ways in which “even and odd divide” is not one phenomenon, the sign of a determinant among them. That is the earlier occupant. What is added is counting the decompositions by length and finding 16 and 640, confirming that even lengths give 0, confirming that three counts agree in parity across all 24 permutations, and placing the separator on length versus parity.

1. Notation and the Question

Setting: a rearrangement of n things is a permutation, and an operation exchanging just two of them is a transposition. Every permutation is a product of transpositions, but not in one way.

$$\text{inv}(p) = \#\{(i, j) : i < j, p_i > p_j\}, \quad \text{sgn}(p) = (-1)^{\text{inv}(p)} = (-1)^{n-c(p)}$$

Here $c(p)$ is the number of cycles. This paper takes $n = 4$ and the four-cycle $(1, 2, 3, 0)$.

Question: what stays the same when the steps differ?

2. Neither Procedure nor Length Is Unique

Computation 2.1: with the 6 transpositions available, the ways of reaching $(1, 2, 3, 0)$ were counted by length.

length	number of decompositions
1	0
2	0
3	16
4	0
5	640
6	0

Observation 2.2

this is the core. Neither procedure nor length is unique. The shortest is 3, but 5 also works and is forty times as common: as with routes to one place, detours are unlimited.

Observation 2.3

and yet no even length works. Lengths 1, 2, 4 and 6 all give 0: the length is undetermined while the parity of the length is fixed.

Observation 2.4

that “fixed” is a counted result. A 0 deserves doubt, but here the odd side gives 16 and 640, so the search was not at fault: the same search hits on odd lengths and never on even ones.

3. Three Counts Return the Same Parity

Computation 3.1: three quantities were taken for (1, 2, 3, 0).

counted	value	parity
shortest decomposition	3	odd
inversion count	3	odd
n minus the number of cycles	$4 - 1 = 3$	odd

Computation 3.2: across all 24 permutations of $n = 4$, the parity of the inversion count and of n minus the number of cycles were compared.

counted	value
permutations	24
disagreements in parity	0
even permutations	12
their share	0.500000

Observation 3.3

not one disagreed. One counts reversals of order, the other counts loops, and nothing in between is shared, so “the sign” is not another name for any one of these counts.

Observation 3.4

even permutations are exactly half. 12 of 24, a share of 0.500000, because multiplying by one transposition matches evens to odds one for one.

4. Why a Determinant Can Be Written

Computation 4.1: for a 4×4 matrix the determinant was obtained two ways.

method	value
signed sum over all 24 permutations	49.500000000
elimination (no signs used)	49.500000000
difference	7.105×10^{-15}
with two rows exchanged	−49.500000000

Observation 4.2

without a single-valued sign the formula cannot be written. The Leibniz form attaches ± 1 to each permutation, and if that ± 1 changed with the decomposition the value would not be defined: the “0 ways at even length” counted in Section 2 is what holds this formula up.

Observation 4.3

exchanging two rows only flips the sign. The two values sum to 0.000×10^0 : a row exchange is one transposition, so every permutation’s parity flips at once.

5. The Separator

Observation 5.1

the separator is whether one asks about the length of the procedure or about its parity. Length can be inflated by detours; parity cannot be changed by anything: the first is about the route, the second a property of the destination.

Observation 5.2

placed beside Paper 601.601 showed that half the positions of the 15 puzzle are unreachable, using the parity of the arrangement. There the shape of the board decides whether parity bites; here parity itself is counted. One uses it, the other holds it up. Paper 247 counted eight ways in which “even and odd divide” is not one phenomenon, the sign of a determinant among them. That is the earlier occupant.

Observation 5.3

what is not said. Group theory, real rearrangements and puzzles are not treated. What is shown is that the undetermined and the determined live inside the very same procedure.

6. Census and Closing

1. **16 ways at length 3** (Computation 2.1): **the shortest.**
2. **640 ways at length 5** (Computation 2.1): **forty times; the core.**
3. **0 ways at even length** (Computation 2.1): **the search hits on odd.**
4. **inversions 3; n minus cycles 3** (Computation 3.1): **the same parity.**
5. **0 disagreements over 24** (Computation 3.2): **nothing in between is shared.**
6. **12 even permutations** (Computation 3.2): **a share of 0.500000.**
7. **signed sum 49.500000000** (Computation 4.1): **over 24 permutations.**
8. **elimination 49.500000000** (Computation 4.1): **difference 7.105×10^{-15} .**
9. **rows exchanged: -49.500000000** (Computation 4.1): **one transposition.**
10. **length, or parity** (Observation 5.1): **the separator; Papers 601, 247.**

Closing: The procedure is not determined, but its parity is. Building the four-cycle from swaps admits 16 decompositions of length 3 and 640 of length 5, the longer forty times as common, because detours are unlimited. And yet no even length works at all, and that parity agrees with the parity of the inversion count and of n minus the number of cycles, with no disagreement across all 24 permutations of $n = 4$, because one count reverses order and the other counts loops, sharing nothing in between. This uniqueness is what lets a determinant be written: the signed sum 49.500000000 and elimination 49.500000000 agree to 7.105×10^{-15} . Placed among the earlier papers: beside Paper 601, which uses parity where this paper holds it up, and opening one of the eight divisions Paper 247 laid out. To be honest: only $n = 4$ and lengths up to 6 were counted; larger cases were not.

References

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