

Whether You Can Win Is Settled by Painting Backwards; How Many Moves It Takes Is Not in That Label

— In a game where **1, 3 or 4** stones may be taken, painting positions **0 to 1000** backwards once each gives **286** losing positions, exactly the ones congruent to **0 or 2** modulo **7** (no disagreement in either direction) — the separator is whether one asks whether you can win or in how many moves — [Paper 846]

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— *From thinking to syncing.* —

Abstract

Solving a game is imagined as reading forwards from the opening. This paper counts that painting from the end is cheaper, and that something is still missing. No new theorem or law is claimed. First, painting backwards settles every position: looking at each of 0 to 1000 once gives **286** losing positions (Section 2). Second, they fall into a pattern: exactly those congruent to **0 or 2** modulo **7**, with no disagreement in either direction (Section 2). Third, the pattern cannot be read off the allowed moves: the moves are **1, 3 and 4**, and the period is **7** (Section 2). Fourth, and this is the core. The label “win” carries no count of moves: within it the number of moves takes **286** distinct values, from **1 to 571** (Section 3). Fifth, searching forwards revisits the same positions: at position **25**, **229970** positions forwards against **26** backwards, a ratio of **8845.000000** (Section 4). Sixth, the separator is whether one asks whether you can win or in how many moves: the second must be carried while painting, or it cannot be recovered afterwards (Section 5). **Keywords:** retrograde analysis, subtraction game, losing position, period, move count, exhaustive enumeration.

Scope of this paper (scope note): No new theorem or law is claimed: settling positions by working back from the end (retrograde analysis), the periodicity of losing positions in a one-heap subtraction game, and the Sprague–Grundy theory, are all known. Real games are not discussed: strength at chess or go, and actual play, are not entered. The periodicity is not proved: that the period is 7 is confirmed by enumeration, not proved; the Sprague–Grundy theory is only named. The move set is not generalised: only 1, 3, 4 is examined; other sets are not treated. Both players are taken to play best: misreadings and time limits are not treated. The size is not generalised: positions above 1000 were not counted. Relation to earlier papers: Paper 602 showed that who wins at Nim is decided by addition without carries. This paper does not sit in that seat but stands beside it: there several heaps and a closed formula, here one heap with no formula and nothing to do but paint. Paper 300 showed that whether two things share a root is decidable: the win-or-lose label and the number of moves are distinct roots by that criterion. What is added is painting to 1000 and confirming the congruence in both directions, counting 286 distinct move counts inside one label, measuring the ratio between forward and backward position counts, and placing the separator on whether versus how many.

1. Notation and the Question

Setting: there are n stones. Two players alternately take 1, 3 or 4; whoever cannot move loses.

$$\text{win}(n) = \bigvee_{m \in \{1,3,4\}, m \leq n} \neg \text{win}(n - m), \quad \text{win}(0) = \text{false}$$

Filling in from 0 upwards settles every position with one visit each. Alongside it, the number of moves to the finish under best play is carried.

Question: what does the finished labelling answer, and what does it not?

2. Painting Backwards

Computation 2.1: positions 0 to 1000 were painted. The first twenty losing positions are 0, 2, 7, 9, 14, 16, 21, 23, 28, 30, 35, 37, 42, 44, 49, 51, 56, 58, 63 and 65.

counted	value
losing positions	286
positions congruent to 0 or 2 modulo 7	286
losing positions outside the congruence	0
positions in the congruence that win	0

Observation 2.2

neither direction disagrees. Both “losing implies congruent” and “congruent implies losing” were counted, and both give 0: counting one direction alone would miss counterexamples of the other.

Observation 2.3

the period 7 cannot be read off the moves. The moves are 1, 3 and 4, and neither their sum nor their largest is 7. It is a number that appears only once the painting is done.

Observation 2.4

the ends are plain. Position 0 loses because nothing can be taken, and 1, 3, 4 win because one move reaches 0: the first few can be checked by hand, and from there it is enumeration.

3. One Label, Many Move Counts

Computation 3.1: alongside the labelling, the number of moves to the finish under best play was carried.

position	label	moves to the finish
10	win	5
20	win	11
50	win	29
100	lose	58
500	win	285
1000	win	571

Observation 3.2

this is the core. The label carries no count. Among winning positions the move counts run from 1 to 571 and take 286 distinct values: all of that is folded into the single word “win”.

Observation 3.3

the count cannot be added afterwards. Looking only at the finished labels, no position’s move count can be recovered: the count is a quantity carried during the painting, not a function of the labels.

4. Searching Forwards Revisits

Computation 4.1: the number of positions visited by a forward search without memory was counted.

position	forwards	backwards	ratio
10	168	11	15.272727
15	1869	16	116.812500
20	20736	21	987.428571
25	229970	26	8845.000000

Observation 4.2

the ratio climbs steeply with the position. At position 25 it is 8845, because a forward search reaches the same position by many routes and reads it again each time.

Observation 4.3

the backward count is just the number of positions. Up to position n it visits $n + 1$, because nothing painted is ever painted twice.

5. The Separator

Observation 5.1

the separator is whether one asks whether you can win or in how many moves. The label answers the first and not the second: within the same painting, one is an output and the other vanishes unless it is made one.

Observation 5.2

placed beside Paper 602.602 showed that who wins at Nim is decided by addition without carries. There several heaps and a closed formula giving the answer directly; here one heap with no formula and nothing to do but paint. Both answer “can you win”, by different means. Paper 300 showed that whether two things share a root is decidable: label and move count are distinct roots by that criterion.

Observation 5.3

what is not said. Real games, a proof of the period and other move sets are not treated. What is shown is that a cheap way of solving need not answer everything one wants to ask.

6. Census and Closing

1. **286 losing positions** (Computation 2.1): **up to 1000.**
2. **congruent to 0 or 2 modulo 7** (Computation 2.1): **0 either way.**
3. **the period is not in the moves** (Observation 2.3): **it appears on painting.**
4. **286 distinct move counts** (Observation 3.2): **the core.**
5. **from 1 to 571** (Computation 3.1): **folded into one word.**
6. **the count cannot be added later** (Observation 3.3): **not a function of labels.**
7. **229970 forwards at position 25** (Computation 4.1): **without memory.**
8. **26 backwards** (Computation 4.1): **ratio 8845.000000.**
9. **backwards equals the positions** (Observation 4.3): **never twice.**
10. **whether, or how many** (Observation 5.1): **the separator; Papers 602, 300.**

Closing: Whether you can win is settled by painting backwards; how many moves it takes is not in that label. Painting positions 0 to 1000 of the game where 1, 3 or 4 stones may be taken gives 286 losing positions, exactly those congruent to 0 or 2 modulo 7, with no disagreement in either direction: and the period 7 is a number that appears only once the painting is done, not one readable from the moves 1, 3, 4. Yet within the single label “win” the number of moves takes 286 distinct values from 1 to 571, and none of them can be recovered from the labels alone, because the count is carried during the painting and is not a function of the labels. Painting is also cheap: at position 25 a forward search visits 229970 positions against 26 backwards, a ratio of 8845.000000. Placed among the earlier papers: beside Paper 602’s “a closed formula gives the answer directly”, as the side with no formula, and treating label and move count as distinct roots by the criterion of Paper 300. To be honest: only the move set 1, 3, 4 was examined, and the period 7 was not proved.

References

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