

What Throws a Pendulum Off When It Is Pushed Is Where It Is Pushed, More Than How Hard

— at the same position the error reverses between the down-swing and the up-swing, and pushing at the bottom removes the first-order error — the separator is whether one asks how hard the push is or where it lands —

[Paper 856]

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Abstract

A pendulum clock keeps going because its escapement pushes it once each cycle. This paper counts how that push disturbs the period. No new theorem or law is claimed. First, the same push disturbs differently at different positions: -1.262623×10^{-4} at 0.25 of the amplitude and -4.807963×10^{-4} at 0.95, a ratio of **3.807916** (Section 2). Second, and this is the core. Pushing at the bottom removes the error: at the bottom the closed form gives 0 and integration gives an error of order 10^{-16} (Section 2). Third, down-swing and up-swing reverse the sign: at 0.50 of the amplitude, a push on the way down shortens the period and on the way up lengthens it (Section 2). Fourth, two ways of computing agree: integration with step 10^{-5} and the closed-form phase agree to within 5×10^{-8} at all five positions (Section 3). Fifth, small errors accumulate: pushing at 0.50 of the amplitude every cycle accumulates **−21.825193** periods over 86400 periods (Section 3). Sixth, the separator is whether one asks how hard the push is or where it lands: a good escapement depends less on how much it pushes than on where (Section 4). **Keywords:** escapement, pendulum, isochronism, phase, Airy's condition, model.

Scope of this paper (scope note): No new theorem or law is claimed: that an escapement's impulse disturbs the period, that the disturbance vanishes to first order when the impulse is given at the bottom of the swing (Airy's condition), and that the phase of an impulse advances or retards the period, are all known. No real clock is claimed: a period of 1, an amplitude of 1 and a push of 0.02 are model values. The mechanism's shape is not entered: the shape of the pallets and escape wheel and how the impulse is delivered are not treated. Circular error is removed: to separate out amplitude dependence the pendulum is taken as linear; the effect of amplitude belongs to Paper 396. No dissipation is included: friction and air resistance are not included. Relation to earlier papers: Paper 396 showed that putting a number on a pendulum's "small angle" costs 164.78 seconds a day at ten degrees, and its scope note left open "clocks are not designed: escapements and temperature compensation are not the subject". This paper sits in that seat; there is no earlier occupant. Paper 488 showed that determining a curve backwards from isochronism leaves only one, likewise leaving the escapement open. There the isochronous curve, here the push that breaks isochronism. What is added is laying out the period error for five push positions, checking integration against the closed form, measuring the sign reversal between down-swing and up-swing, and placing the separator on how hard versus where.

1. Notation and the Question

Setting: a linear pendulum with angular frequency $\omega = 2\pi$ (period 1) and amplitude $A = 1$ receives $\Delta v = 0.02$ added to its angular velocity as it passes x_0 on the down-swing. The period error of that cycle follows from the change in phase.

$$\Delta T = -\frac{1}{\omega} \left[\text{atan2} \left(\sqrt{A^2 - x_0^2} + \frac{\Delta v}{\omega}, x_0 \right) - \text{atan2} \left(\sqrt{A^2 - x_0^2}, x_0 \right) \right] \approx -\frac{x_0 \Delta v}{\omega^2 A^2}$$

The approximation on the right is first order and vanishes at $x_0 = 0$ (the bottom). Separately, one cycle is integrated by Runge--Kutta with step 10^{-5} .

Question: does a push disturb the period because of how hard it is, or where it lands?

2. Push Position and Error

Computation 2.1: with the push fixed at 0.02, the down-swing position was varied.

push position (fraction of amplitude)	period error (closed form)
0.00 (bottom)	0
0.25	-1.262623×10^{-4}
0.50	-2.526064×10^{-4}
0.75	-3.791554×10^{-4}
0.95	-4.807963×10^{-4}

Observation 2.2

with the same strength, pushes nearer the end disturb more. The ratio of 0.95 to 0.25 is 3.807916: the shift in phase is proportional to the displacement at the moment of the push.

Observation 2.3

this is the core. Pushing at the bottom removes the error. The closed form gives 0 and integration 1.110×10^{-16} : at the bottom the displacement is 0, so changing the speed does not move the phase.

Computation 2.4: at 0.50 of the amplitude, a push was given on the down-swing and on the up-swing.

push	period error (integration)
on the down-swing	-0.000252597
on the up-swing	+0.000252625

Observation 2.5

at the same position, the sign reverses. Down-swing shortens, up-swing lengthens: pushing on the side heading toward the bottom or away from it swaps phase advance and retard.

3. Two Computations, and Accumulating Error

Computation 3.1: the closed form, the first-order approximation and integration were laid side by side.

position	closed form	first order	integration
0.25	-1.262623×10^{-4}	-1.266515×10^{-4}	-1.262460×10^{-4}
0.50	-2.526064×10^{-4}	-2.533030×10^{-4}	-2.525972×10^{-4}
0.75	-3.791554×10^{-4}	-3.799544×10^{-4}	-3.791485×10^{-4}
0.95	-4.807963×10^{-4}	-4.812756×10^{-4}	-4.807876×10^{-4}

Observation 3.2

integration and the closed form agree within 5×10^{-8} . One steps through time; the other reads the phase angle directly: the first-order approximation overstates the magnitude by 0.002757.

Observation 3.3

a suspicious run time was also checked. The whole check finished in 0.4 seconds, so whether the 100000-step integration really ran was counted separately: 100000 steps in 0.096 seconds: varying the step through 10^{-3} , 10^{-4} and 10^{-5} , the error converged at first order.

Computation 3.4: pushing at 0.50 of the amplitude every cycle accumulates -21.825193 periods over 86400 periods.

Observation 3.5

an error of order 10^{-4} per cycle grows large. In a one-second model it exceeds 21 seconds a day: pushing at the bottom removes the first-order source of that accumulation.

4. The Separator

Observation 4.1

the separator is whether one asks how hard the push is or where it lands. Without changing strength, position moves the error from 0 to -4.807963×10^{-4} : a good escapement depends less on how much it pushes than on where.

Observation 4.2

sitting in the seat of Paper 396.396 showed that the “small angle” costs 164.78 seconds a day at ten degrees and left the escapement open. There amplitude breaks isochronism; here the push does. Amplitude was removed with a linear pendulum so that only the push is counted. Paper 488 showed that the isochronous curve is unique: the curve side and the push side of the same isochronism.

Observation 4.3

what is not said. Real clocks, escapement shapes, amplitude error and friction are not treated. What is shown is that the push’s error is set by position and vanishes to first order at the bottom.

5. Census and Closing

1. bottom: error 0 (Computation 2.1): the core.
2. 0.25: -1.262623×10^{-4} (Computation 2.1): nearer the end, more.
3. 0.95: -4.807963×10^{-4} (Computation 2.1): large.
4. ratio **3.807916** (Observation 2.2): proportional to displacement.
5. down shortens, up lengthens (Computation 2.4): sign reverses.
6. two computations within 5×10^{-8} (Observation 3.2): independent.
7. first order overstates by 0.002757 (Observation 3.2): the approximation’s miss.
8. 100000 steps in 0.096 s (Observation 3.3): checked, not assumed.
9. a day gives **-21.825193** periods (Computation 3.4): it accumulates.
10. how hard, or where (Observation 4.1): the separator; Papers 396, 488.

Closing: What throws a pendulum off when it is pushed is where it is pushed, more than how hard. Adding 0.02 to a unit-period pendulum's angular velocity gives a period error of 0 at the bottom, -1.262623×10^{-4} at 0.25 of the amplitude and -4.807963×10^{-4} at 0.95, growing with the displacement at the push: the phase shift is set by the displacement at that instant, and at the bottom the displacement is 0. At the same position the error reverses between down-swing and up-swing, and integration and the closed form agree within 5×10^{-8} at all five positions: one steps through time and the other reads the phase directly, sharing nothing in between. Pushing at 0.50 of the amplitude accumulates -21.825193 periods over 86400, and pushing at the bottom removes that first-order source. Placed among the earlier papers: sitting in the escapement seat Paper 396 left open, and beside 396's amplitude and Paper 488's isochronous curve, counting the push that breaks isochronism. To be honest: the pendulum is linear, so amplitude effects and friction are excluded, and the push of 0.02 is a model value.

References

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