

Splitting Into Groups Comes in Two Kinds, and Arranging Each Group in a Cycle Changes Even the Most Common Number of Groups

— Splitting 7 items into groups can be done in **877** ways (the Bell number), but in **5040** ways (**7!**) if each group is arranged in a cycle, a ratio of **5.746864** — the separator is whether one looks only at the contents of each group or also at the order around it — [Paper 870]

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Abstract

“How many ways can 7 people be split into three groups?” looks like one question. This paper checks, by counting everything, that allowing an order around each group splits the count into two and changes even the most common number of groups. No new theorem or law is claimed. First, there are two ways to count splittings: the number of plain splittings (Stirling numbers of the second kind) and of splittings with each group in a cycle (Stirling numbers of the first kind) were enumerated and matched with both recurrences for every number of groups with $n \leq 7$ (Section 2). Second, the totals differ in magnitude: for 7 items the plain total is the Bell number 877 and the cycle total is $7! = 5040$, a ratio of **5.746864** (Section 2). Third, and this is the core. Even the most common number of groups changes: when only splitting, 4 groups is most common with 350 ways; when arranging in cycles, 2 groups is most common with 1764 ways (Section 3). Fourth, the mean number of groups differs too: for a random choice the mean is **3.720639** when only splitting and 2.592857 with cycles, the latter matching the harmonic number H_7 within 10^{-12} (Section 3). Fifth, the number of cycles decides parity: a permutation split into cycles is even when 7 minus the number of cycles is even, and there are exactly half of 5040, namely 2520 (Section 4). Sixth, the separator is whether one looks only at the contents of each group or also at the order around it: allowing an order around each group multiplies the arrangements of large groups, favouring fewer, larger groups (Section 5). **Keywords:** Stirling numbers, Bell numbers, cycle decomposition, harmonic number, even permutation.

Scope of this paper (scope note): No new theorem or law is claimed: the two kinds of Stirling numbers and their recurrences, the totals being Bell numbers and factorials, the mean number of cycles being a harmonic number, and the number of cycles deciding the parity of a permutation, are all known. No proofs are entered: generating functions and formulas for large n are not entered. Size is not generalised: $n \geq 8$ was not enumerated. Real groupings are not treated: forming teams or seating is not the subject. Relation to earlier papers: Paper 841 showed that splitting into transpositions is not determined, yet parity is. This paper stands beside it. Here the splittings of permutations into cycles are counted, and the number of cycles deciding parity is checked with the 2520 even permutations. What is added is enumerating both kinds for $n \leq 7$ and matching the recurrences, measuring the ratio 5.746864 of the totals, measuring that the most common number of groups splits into 4 and 2, and placing the separator on contents versus order around the group.

1. Notation and the Question

Setting: n labelled items are split into k nonempty groups. The order of the groups is not distinguished. Let $S(n, k)$ (Stirling numbers of the second kind) count splittings ignoring order within groups, and $c(n, k)$ (unsigned Stirling numbers of the first kind) count splittings in which each group is arranged in a cycle. The latter is also the number of permutations of n with exactly k cycles.

$$S(n, k) = kS(n-1, k) + S(n-1, k-1), \quad c(n, k) = (n-1)c(n-1, k) + c(n-1, k-1),$$

$$\sum_k S(n, k) = B_n, \quad \sum_k c(n, k) = n!$$

$S(n, k)$ was counted by generating all group-assignment sequences, and $c(n, k)$ by generating all permutations and counting cycles; both were compared with the recurrences for every k with $n \leq 7$.

Question: how much does the number of ways to form k groups change depending on whether order within groups is allowed?

2. Two Ways to Count

Computation 2.1: for $n = 7$, both numbers were counted for each k .

k	only splitting $S(7, k)$	in cycles $c(7, k)$	ratio
1	1	720	720.000000
2	63	1764	28.000000
3	301	1624	5.395349
4	350	735	2.100000
5	140	175	1.250000
6	21	21	1.000000
7	1	1	1.000000

Observation 2.2

the totals differ in magnitude. The plain totals are the Bell numbers 1, 2, 5, 15, 52, 203, 877, the cycle totals are the factorials 1, 2, 6, 24, 120, 720, 5040, and at $n = 7$ the ratio is 5.746864: the enumerated counts matched both recurrences for every k with $n \leq 7$.

Observation 2.3

with small groups, the two coincide. At $k = 6$ and $k = 7$ the ratio is 1.000000: a group of size 1 or 2 has only one cyclic order. The gap opens on the small- k side, where large groups form.

3. The Most Common Number of Groups

Computation 3.1: for $n = 7$, the most common number of groups and the mean number of groups for a random choice were found.

quantity	only splitting	in cycles
most common number of groups	4 (350 ways)	2 (1764 ways)
mean number of groups	3.720639	2.592857

Observation 3.2

this is the core. Even the most common number of groups changes. When only splitting, 4 groups is the most common; when arranging in cycles, 2 groups is, and the ratio of the means is 1.434957: a group of size m has $(m - 1)!$ cyclic orders, so splittings with fewer, larger groups win on count. The cycle mean 2.592857 matched the harmonic number $H_7 = 1 + \frac{1}{2} + \cdots + \frac{1}{7}$ within 10^{-12} .

4. Number of Cycles and Parity

Computation 4.1: the permutations of 7 were split into cycles, and those with 7 minus the number of cycles even (even permutations) were counted.

Observation 4.2

the number of cycles decides parity. There are 2520 even permutations, exactly half of 5040: a cycle of length m is a product of $m - 1$ transpositions, so the parity of the number of transpositions equals the parity of 7 minus the number of cycles. Even when splitting into transpositions is not determined, the number of cycles is.

5. The Separator

Observation 5.1

the separator is whether one looks only at the contents of each group or also at the order around it. Looking only at contents, a moderate number of groups, 4, wins: looking also at the order around each group, the arrangements inside large groups push the count up and a small number of groups, 2, wins. The same “split 7 items into groups” changes its most common shape according to what is distinguished.

Observation 5.2

placed beside Paper 841.841 showed that splitting into transpositions is not determined, yet parity is. Here we counted splittings of permutations into cycles, and checked that the number of cycles decides parity, with even permutations exactly half.

Observation 5.3

what is not said. Proofs through generating functions or asymptotics, enumeration for $n \geq 8$, and real groupings are not treated. What is shown is that allowing an order around each group changes both the count and the most common number of groups.

6. Census and Closing

1. $S(n, k)$ and $c(n, k)$ (Section 1): **only splitting, and in cycles.**
2. **recurrences match enumeration** (Observation 2.2): $n \leq 7$.
3. **totals 877 and 5040** (Observation 2.2): **Bell number and factorial.**
4. **ratio 5.746864** (Observation 2.2): $n = 7$.
5. **factor 28 for two groups** (Computation 2.1): **63 and 1764.**
6. **equal at $k = 6, 7$** (Observation 2.3): **small groups have one order.**
7. **most common: 4 and 2 groups** (Computation 3.1): **the core.**
8. **means 3.720639 and 2.592857** (Computation 3.1): **the latter harmonic.**
9. **2520 even permutations** (Observation 4.2): **exactly half.**
10. **contents, or order around the group** (Observation 5.1): **the separator; Paper 841.**

Closing: Splitting into groups comes in two kinds, and arranging each group in a cycle changes even the most common number of groups. 7 items can be split into groups in 877 ways, but in 5040 ways with each group in a cycle, a ratio of 5.746864, because a group of size m has $(m - 1)!$ cyclic orders, and this growth is largest for large groups. So the most common number of groups is 4 when only splitting and 2 in cycles, and the means are 3.720639 and 2.592857: the number of cycles decides the parity of a permutation, and even permutations are exactly half, 2520. Placed among the earlier papers: beside Paper 841's "only parity is determined", we counted that parity through the number of cycles and placed how allowing cycles changes the count of splittings itself. To be honest: enumeration went only up to $n \leq 7$, and the recurrences and the harmonic-number match are known.

References

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