

For Vertex Cover, the Twice-Guaranteed Method Loses to Greed on Ordinary Graphs, and Greed Exceeds Twice on a Built One

— When choosing a small set of vertices touching every edge (a vertex cover), taking edges one by one and adding both ends is guaranteed never to exceed twice the optimum — the separator is whether one asks about the average ratio or the worst ratio — [Paper 875]

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September 2026

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Abstract

It is easy to assume that a method guaranteed never to exceed twice the optimum is better than one with no guarantee. This paper compares two methods for vertex cover with the optimum on both random and built graphs. No new theorem or law is claimed. First, the guarantee holds: taking edges one by one and adding both ends never exceeded twice the optimum on any of 300 random graphs with 12 vertices, and gave exactly twice on 6 disjoint edges (Section 2). Second, on ordinary graphs, unguaranteed greed wins: at edge probability 0.2, the guaranteed method averages **1.680857** times the optimum and greed, choosing vertices with the most edges, averages **1.030833** (Section 2). Third, and this is the core. On a built graph, greed exceeds twice: with 30 vertices on the left and blocks of sizes 2 to 30 on the right, greed takes 81 vertices against an optimum of 30, a ratio of **2.700000** (Section 3). Fourth, on the same graph, the guaranteed method stays at twice: 60 vertices, a ratio of 2.000000, within its guarantee (Section 3). Fifth, built small, greed still wins: with 12 vertices on the left, greed takes 23 (1.916667 times) and the guaranteed method 24 (2.000000 times) (Section 3). Sixth, the separator is whether one asks about the average ratio or the worst ratio: the guarantee is a promise about the worst ratio, not about performance on ordinary inputs (Section 4). Keywords: vertex cover, approximation ratio, greedy algorithm, maximal matching, worst-case ratio.

Scope of this paper (scope note): No new theorem or law is claimed: that finding a minimum vertex cover is generally hard, that taking both ends of a maximal matching never exceeds twice the optimum, and that greed by degree has no constant-factor guarantee, are all known. Hardness proofs are not entered: the theory of how well one can approximate is not entered. Nothing is designed, where to monitor a real network is not stated. Optima only where they can be checked: random graphs had 12 vertices and were checked exhaustively; the built graphs' optima were confirmed two ways, by the left vertices covering all edges and by the size of a maximum matching. Relation to earlier papers: Paper 348 showed that on the same graph connecting is cheap and ordering is not, and in its scope note did not enter approximation ratios. This paper takes that seat (no one has taken it before), counting approximation ratios not for touring but for vertex cover. Paper 633 showed that paying largest coins first uses the fewest coins in yen but one too many at 48 pence in old British coins. This paper stands beside it too. There, one input where greed loses was shown. Here, greed wins on average and exceeds twice on a built input. What is added is comparing the averages and maxima of both methods with the optimum on 300 random graphs, measuring greed at 2.700000 times on a built graph, confirming the optimum two ways, and placing the separator on average ratio versus worst ratio.

1. Notation and the Question

Setting: a set of vertices containing at least one end of every edge is a vertex cover. We want as few vertices as possible. Two methods are compared.

guaranteed : take an edge with neither end chosen, add both ends, greedy : add the vertex with the most remaining edges

In the former, the chosen edges share no ends, so every cover contains one end of each, and the result never exceeds twice the optimum. The latter has no such guarantee. The ratio is the number of chosen vertices divided by the optimum.

Question: is a method guaranteed within twice better than one without a guarantee?

2. Random Graphs

Computation 2.1: with 12 vertices and varying edge probability, 100 graphs each were made, the optimum found exhaustively, and both methods' ratios computed.

edge probability	guaranteed mean	guaranteed max	greedy mean	greedy max
0.2	1.680857	2.000000	1.030833	1.250000
0.35	1.442643	2.000000	1.028857	1.400000
0.5	1.386786	1.714286	1.019464	1.142857

Observation 2.2

on ordinary graphs, unguaranteed greed wins. At edge probability 0.2, the guaranteed method averages 1.680857 and greed 1.030833: the guaranteed method never exceeded twice on any of 300 graphs, and its maximum reached exactly 2.000000. On 6 disjoint edges it takes 12 vertices against an optimum of 6, exactly twice.

3. A Built Graph

Computation 3.1: k vertices were placed on the left, and for each size $i = 2, \dots, k$, $\lfloor k/i \rfloor$ right vertices were each joined to i left vertices. The k left vertices cover every edge and a maximum matching has k edges, so the optimum is k .

k	vertices	edges	optimum	greedy	guaranteed
12	35	115	12	23 (1.916667)	24 (2.000000)
30	111	732	30	81 (2.700000)	60 (2.000000)

Observation 3.2

this is the core. On a built graph, greed exceeds twice. At $k = 30$, greed chooses 81 vertices, 2.700000 times the optimum: a right vertex in a large block always has more remaining edges than any left vertex, so greed keeps choosing right vertices and fills with them what 30 left vertices would have covered. On the same graph, the guaranteed method takes 60, within its guarantee.

Observation 3.3

built small, greed still wins. At $k = 12$, greed takes 23 and the guaranteed method 24, greed one fewer: with the same construction, greed overtakes the guaranteed method only once the blocks are made larger.

4. The Separator

Observation 4.1

the separator is whether one asks about the average ratio or the worst ratio. The guaranteed method promises about the worst case: no input makes it exceed twice. It promises nothing about how good it is on ordinary inputs. Greed has no worst-case promise, yet averages 1.030833 on random graphs, close to optimal. Which is better switches depending on whether one asks about the average or the worst.

Observation 4.2

taking the seat of Paper 348, and placed beside Paper 633. 348 did not enter approximation ratios in its scope note. Here, approximation ratios for vertex cover are counted, setting guaranteed ratios beside actual ones. 633 showed that paying largest coins first uses one coin too many at 48 pence in old British coins. Here, greed wins on average and exceeds twice on a built graph.

Observation 4.3

what is not said. Hardness of approximation, how to monitor real networks, and optima of random graphs with 13 or more vertices are not treated. What is shown is that the average ratio and the worst ratio are different quantities, and they switch which method wins.

5. Census and Closing

1. guaranteed within **2** times (Section 1): **chosen edges share no ends.**
2. all **300** within twice (Observation 2.2): **the guarantee holds.**
3. exactly twice on **6** disjoint edges (Observation 2.2): **12 against 6.**
4. guaranteed mean **1.680857** (Computation 2.1): **edge probability 0.2.**
5. greedy mean **1.030833** (Computation 2.1): **close to optimal.**
6. greed takes **81** on the built graph (Computation 3.1): **optimum 30.**
7. ratio **2.700000** (Observation 3.2): **the core.**
8. guaranteed takes **60** on the same graph (Computation 3.1): **twice.**
9. **23** against **24** at $k = 12$ (Observation 3.3): **greed still wins.**
10. average ratio, or worst ratio (Observation 4.1): **the separator; the seat of Paper 348; Paper 633.**

Closing: For vertex cover, the method guaranteed within twice loses to greed on ordinary graphs, and greed exceeds twice on a built graph. On random graphs with 12 vertices and edge probability 0.2, the guaranteed method averages 1.680857 and greed by most edges averages 1.030833, because the guaranteed method always adds both ends of an edge, including ends that one alone would have covered. On a graph with 30 vertices on the left and blocks of different sizes on the right, greed takes 81 vertices against an optimum of 30 (2.700000 times), while the guaranteed method stays at 60: the guarantee is a promise about the worst ratio, not about average performance. Placed among the earlier papers: we took the approximation-ratio seat left open in Paper 348's scope note with vertex cover, and placed beside Paper 633's greed an example where average and worst switch the winner. To be honest: random graphs had only 12 vertices, and only one construction was built.

References

1. Garey, M. R., & Johnson, D. S. (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman.
2. Vazirani, V. V. (2001). *Approximation Algorithms*. Springer.
3. Yamagishi, Y. (2026). *On the Same Graph, Connecting Is Cheap and Ordering Is Not* [Paper 348]. YOI: <https://www.yamagishiyuuki.com/yoi/348>; DOI: 10.5281/zenodo.22683555
4. Yamagishi, Y. (2026). *Paying Largest Coins First Always Uses the Fewest Coins in Yen, but One Too Many at 48 Pence in Old British Coins* [Paper 633]. YOI: <https://www.yamagishiyuuki.com/yoi/633>; DOI: 10.5281/zenodo.23014568

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