

Triples Covering Every Pair Exactly Once Exist, up to Fifteen Points, Only When the Count Leaves Remainder One or Three Mod Six

— where none exists, the number of triples $v(v-1)/6$ is not an integer (**3.333333**, **9.333333**, **18.333333**, **30.333333**) or the number of partners $v-1$ of a point is odd — the separator is whether a system is absent by the divisibility conditions or absent even though they hold — [Paper 898]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

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YOI: <https://www.yamagishiyuuki.com/yoi/898>

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Abstract

It is easy to assume that whether points can be grouped into triples so that every pair shares exactly one triple can only be settled by trying each number of points in turn. This paper searches for such triple systems from 3 to 15 points and counts what decides whether they exist. No new theorem or law is claimed. First, only five point counts are found: of 3 to 15 points, triple systems were found only for 3, 7, 9, 13 and 15 (Section 2). Second, and this is the core. Existence is decided by the number of points modulo six: the five found all have remainder 1 or 3, and the eight not found all have remainder 0, 2, 4 or 5 (Section 2). Third, every absence is explained by divisibility: for 5, 8, 11 and 14 points the number of triples is **3.333333**, **9.333333**, **18.333333** and **30.333333**, not an integer, and for 4, 6, 8, 10, 12 and 14 points the number of partners of a point is odd (Section 2). Fourth, when they exist, there are not few: counted with labels, there are 30 systems for 7 points and 840 for 9 (Section 3). Fifth, up to isomorphism each is unique: applying every relabelling to one system found gives exactly 30 and 840 distinct systems, and the relabellings leaving it unchanged number 168 and 432 (Section 3). Sixth, the separator is whether a system is absent by the divisibility conditions or absent even though they hold: up to 15, every absence was by the divisibility conditions, and none was absent with the conditions holding (Section 4). **Keywords:** Steiner triple system, combinatorial design, pairs of points, remainder modulo six, isomorphism, Kirkman.

Scope of this paper (scope note): No new theorem or law is claimed: that collections in which every pair lies in exactly one triple (called Steiner triple systems) exist exactly when the number of points leaves remainder 1 or 3 on division by 6 is a known theorem since Kirkman (1847). That there are 30 for 7 points and 840 for 9, each unique up to isomorphism, is also known. The theorem is not proved: the construction for every point count with remainder 1 or 3 is not reproduced; only 3 to 15 points were searched. Sixteen or more is not searched: counts for larger numbers of points are not treated. Uses are not treated: use in experimental design or tournament scheduling is not treated. Relation to earlier papers: Paper 604 counted that no Latin square of order six has an orthogonal mate, and stated as a known result the side on which Euler's conjecture (no pair for orders leaving remainder 2 on division by 4) fails. This paper stands beside it. There, a remainder condition that held within the computed range was known to fail beyond it. Here, the remainder condition holds for every count up to 15 and is known to hold beyond, as Kirkman's theorem. What is added is searching whether systems exist from 3 to 15 points, laying out that every absence is explained by divisibility, confirming the counts for 7 and 9 points in two ways, and placing the separator on absent by the conditions versus absent despite them.

1. Notation and the Question

Setting: there are v points, and some triples of points are chosen. When every pair lies in exactly one triple, the collection is called a triple system. Each triple covers three pairs, so the number of pairs must be divisible by 3. And the $v - 1$ partners of a point are covered two at a time by the triples containing it, so $v - 1$ must be even.

$$\text{pairs} = \frac{v(v-1)}{2}, \quad \text{triples} = \frac{v(v-1)}{6}, \quad v-1 \text{ even}$$

The search takes the uncovered pair with the smallest labels, tries third points in turn, and backs up when covering fails. For 7 and 9 points every system was counted; for other counts the search ran until one was found or the search was exhausted.

Question: can the existence of a triple system only be settled by trying each number of points?

2. Counts With and Without Systems

Computation 2.1: triple systems were searched for from 3 to 15 points.

points	remainder mod 6	triples	partners	found
3	3	1.000000	2	yes
4	4	2.000000	3	no
5	5	3.333333	4	no
6	0	5.000000	5	no
7	1	7.000000	6	yes
8	2	9.333333	7	no
9	3	12.000000	8	yes
10	4	15.000000	9	no
11	5	18.333333	10	no
12	0	22.000000	11	no
13	1	26.000000	12	yes
14	2	30.333333	13	no
15	3	35.000000	14	yes

Observation 2.2

this is the core. Existence is decided by the number of points modulo six. Systems were found for five counts, 3, 7, 9, 13 and 15, all with remainder 1 or 3: the eight without have a non-integer number of triples (5, 8, 11, 14 points) or an odd number of partners (4, 6, 8, 10, 12, 14 points). Both divisibility conditions hold exactly when the remainder is 1 or 3. Only for 11 points did exhausting the search take more than 20 seconds; the other counts took under 1 second.

3. How Many When They Exist

Computation 3.1: all labelled triple systems were counted for 7 and 9 points. As a separate method, one system was found by a different search, every relabelling of the points (5040 for 7 points, 362880 for 9) was applied, and the distinct systems produced were counted.

points	systems counted	systems from relabelling	relabellings leaving it unchanged
7	30	30	168
9	840	840	432

Observation 3.2

up to isomorphism each is unique. The number of distinct systems produced by relabelling one system equalled the number counted by search. That is, every system is a relabelling of the one found.

$5040 = 30 \times 168$ and $362880 = 840 \times 432$.

4. The Separator

Observation 4.1

the separator is whether a system is absent by the divisibility conditions or absent even though they hold. An absence by the divisibility conditions is known without trying a single triple: an absence with the conditions holding could only be known by trying. Up to 15 points, every absence was of the first kind, and there was none of the second.

Observation 4.2

placed beside Paper 604.604 counted that no Latin square of order six has an orthogonal mate, and stated as a known result that Euler's conjecture, "no pair for orders leaving remainder 2 on division by 4", fails beyond the computed range. Here the remainder condition holds for every count up to 15 points and is known to hold beyond, as Kirkman's theorem. Of two guesses stated by a remainder, one failed and one held.

Observation 4.3

what is not said. A proof of the theorem, a construction for every count with remainder 1 or 3, counts for 16 or more points, and use in experimental design or tournament scheduling are not treated. What is shown is that, up to 15 points, the existence of a triple system is decided by the remainder modulo six, and every absence is explained by divisibility.

5. Census and Closing

1. triples $v(v-1)/6$ and partners $v-1$ (Section 1): two divisibility conditions.
2. found for 3, 7, 9, 13, 15 (Computation 2.1): only five.
3. remainder 1 or 3 (Observation 2.2): the core.
4. **3.333333, 9.333333** (Computation 2.1): not integers for 5 and 8 points.
5. **18.333333, 30.333333** (Computation 2.1): nor for 11 and 14.
6. odd number of partners (Observation 2.2): 4, 6, 8, 10, 12, 14 points.
7. 30 for 7 points (Computation 3.1): 30 by relabelling too.
8. 840 for 9 points (Computation 3.1): 840 by relabelling too.
9. 168 and 432 (Observation 3.2): unique up to isomorphism.
10. absent by the conditions, or despite them (Observation 4.1): the separator; Paper 604.

Closing: Whether triples can cover every pair exactly once is decided by the number of points modulo six: up to fifteen, they exist only for remainders one and three. Searching from 3 to 15 points found systems only for 3, 7, 9, 13 and 15: the eight without had a non-integer number of triples or an odd number of partners per point, all known to be absent before trying any triple. There were 30 systems for 7 points and 840 for 9, relabelling gave the same numbers, and each is unique up to isomorphism. Here the remainder condition did not just predict; it was the reason for every absence. Placed among the earlier papers: beside the Latin squares of Paper 604, where a guess stated by a remainder failed beyond the computed range, we placed triples for which the remainder condition holds. To be honest: the search went only up to 15 points, and that the condition holds for every count rests on Kirkman's known theorem.

References

1. Colbourn, C. J., & Dinitz, J. H. (Eds.) (2007). *Handbook of Combinatorial Designs* (2nd ed.). Chapman & Hall/CRC.

2. Kirkman, T. P. (1847). On a problem in combinations. *Cambridge and Dublin Mathematical Journal*.
3. Yamagishi, Y. (2026). *No Latin Square of Order Six Has an Orthogonal Mate* [Paper 604]. YOI:
<https://www.yamagishiyuuki.com/yoi/604>; DOI: 10.5281/zenodo.23014438

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