Carrying a Polarisation Around a Closed Path Shifts Its Phase by Half the Enclosed Solid Angle

— enclosing a hemisphere gives a phase of π and flips the sign, the same half as the Berry phase of a qubit — the separator is whether one measures intensity alone or the phase by interference with a reference — [Paper 906]

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September 2026

— *From thinking to syncing.* —

Abstract

Passing light through a few polarisers and back to its original polarisation, it is easy to assume the light has returned to what it was.This paper follows the state of polarisation as a point on the Poincaré sphere and compares the phase left after a closed path with the solid angle the path encloses.No new theorem or law is claimed. First, the map to the state of polarisation is the Hopf map——the Stokes vectors built from 2000 two-component complex vectors all lie on the unit sphere and do not move, to the order of 10^{-15}, when an overall phase is applied (Section 2). **Second, and this is the core. The phase of a closed path is half the solid angle enclosed——returning through three polarisers $A \to B \to C \to A$, the phase $\arg\langle A|C\rangle\langle C|B\rangle\langle B|A\rangle$ agreed with $-\Omega/2$ of the enclosed solid angle within 10^{-9} for five random triangles (for instance $\Omega = -5.580641$ gives phase $+2.790320$). The reverse path gives $+\Omega/2$, which equals $-(4\pi - \Omega)/2$ modulo 2π** (Section 2). **Third, enclosing a hemisphere flips the sign——turning a linear polarisation through half a turn in N steps and back (once around the equator, $\Omega = 2\pi$) gives a phase of π for every N from 3 to 384** (Section 3). **Fourth, the amplitude returns but the sign does not——the surviving amplitude is $\cos^N(\pi/N)$, rising from 0.125000 at $N = 3$ to 0.987231 at $N = 384$, yet the intensity when overlaid in phase with a reference cancels to 0.000041 at $N = 384$** (Section 3). **Fifth, finer steps converge to the half on any path——around a circle of angular radius 10 degrees near circular polarisation ($\Omega = 0.095456$), the phase moves from -0.039621 at $N = 6$ to -0.047728 at $N = 1536$, within 1.30×10^{-7} of $-\Omega/2$** (Section 4). **Sixth, the separator is whether one measures intensity alone or the phase by interference with a reference——by intensity alone, the finer the steps the more fully the light returns; overlaid with a reference, the hemisphere's phase π remains as darkness. The formula is that of the same two-component vector as a spin-$1/2$ qubit, and shares its root with the $-\Omega/2$ of Paper 68** (Section 5). **Keywords**: polarisation, Poincaré sphere, geometric phase, solid angle, Hopf map, interference.

Scope of this paper (scope note): **No new theorem or law is claimed——that a closed path of polarisations acquires a phase of half the solid angle (Pancharatnam 1956), that this is the same as Berry's phase (1984; Berry 1987), and that the state space of polarisation is a Hopf fibration are all known.No real optical system is claimed——the number, angles and amplitudes of the polarisers are model values, not those of any apparatus.Partial polarisation and loss are not entered——only projections of fully polarised light are treated; partial polarisation, differential absorption and non-adiabatic effects are not included.The origin of the phase is not entered——that the phase is the holonomy of a Hopf fibration is the matter of Paper 68; here only the numbers for light are given.Relation to earlier papers**: **Paper 68 showed that the qubit is a Hopf fibration and that the Berry phase is measurable as the holonomy $-\Omega/2$ of its fibre——the two-component Jones vector used here has the same form as a spin-$1/2$ spinor, and the same function gives the same phase. Since the formula, not merely the number, is the same, this is written as the same root in the sense of Paper 300.Paper 207 showed that there are three tools for writing polarisation and that the Jones vector cannot write unpolarised light——here the closed path is followed on the third of them, the Poincaré sphere.Paper 08 showed that in the Hopf fibration (the map from S^3 to S^2 whose fibre is S^1) the area of the base times the circumference of the fibre is the volume of the whole——here the base is the Poincaré sphere and the fibre is the phase of the polarisation.Paper 18 separated three roots of 2π and took one of them to be the solid angle of a hemisphere——here the hemisphere's solid angle 2π becomes a phase of π.**What is added is **checking the phase of closed paths made of polariser projections against $-\Omega/2$ on random triangles, showing in numbers that on the hemispherical path the amplitude returns while the sign does not, measuring the convergence as the steps are refined**, and **placing the separator on intensity versus interference**.

1. Notation and the Question

Setting: **the state of polarisation is written as the normalised Jones vector $|\psi\rangle = (\cos\frac{\theta}{2}, e^{i\varphi}\sin\frac{\theta}{2})$ corresponding to polar angle θ and azimuth φ on the Poincaré sphere.The Stokes vector is $s = (2\mathrm{Re}\bar{\psi}_0\psi_1, 2\mathrm{Im}\bar{\psi}_0\psi_1, |\psi_0|^2 - |\psi_1|^2)$, which is the Hopf map.A polariser is a projection onto a state, and the amplitude surviving a passage $A \to B \to \cdots \to A$ is the product of the inner products of neighbouring states.**

$$\mathcal{A} = \langle A|Z\rangle \cdots \langle C|B\rangle\langle B|A\rangle, \qquad \gamma = \arg\mathcal{A}$$

The enclosed solid angle Ω of a triangle was obtained in two ways: from $\tan(\Omega/2) = a\cdot(b\times c)/(1 + a\cdot b + b\cdot c + c\cdot a)$ and from the sum of the three interior angles minus π.The intensity when overlaid with a reference (amplitude 1, phase 0) was taken as $|1+\mathcal{A}|^2/4$.

Question: **when light is brought back to its original polarisation through polarisers, has it returned to what it was?**

2. The Hopf Map and Triangular Paths

Computation 2.1: the Hopf map was checked on 2000 random states, and the phase and solid angle of closed paths were obtained for five random triangles.

triangle	phase γ	solid angle Ω (two ways)	$-\Omega/2$	phase of the reverse path
1	$+2.790320$	-5.580641 (excess 5.580641)	$+2.790320$	-2.790320
2	$+1.379210$	-2.758419 (excess 2.758419)	$+1.379210$	-1.379210
3	-0.049433	$+0.098867$ (excess 0.098867)	-0.049433	$+0.049433$
4	$+2.534184$	-5.068369 (excess 5.068369)	$+2.534184$	-2.534184
5	$+1.364392$	-2.728783 (excess 2.728783)	$+1.364392$	-1.364392

> **Observation 2.2**
>
> this is the core. The phase of a closed path is half the solid angle enclosed.All 2000 Stokes vectors lay on the unit sphere, did not depend on the overall phase, and coincided with the point (θ, φ)——the phase of each of the five triangular paths agreed with $-\Omega/2$ within 10^{-9}, and the oriented solid angle agreed with the angular excess. The phase of the reverse path, $+\Omega/2$, equals $-(4\pi - \Omega)/2$ modulo 2π: the full sphere 4π corresponds to one turn of phase, 2π.

3. The Hemispherical Path

Computation 3.1: the phase and amplitude were obtained for the path that turns a linear polarisation through half a turn in N steps and back to the start (once around the equator of the Poincaré sphere, $\Omega = 2\pi$).

N	phase	surviving amplitude $\cos^{N}(\pi/N)$	intensity overlaid in phase with a reference
3	π	0.125000	0.191406
6	π	0.421875	0.083557
12	π	0.659668	0.028956
24	π	0.813665	0.008680
96	π	0.949886	0.000628
384	π	0.987231	0.000041

Observation 3.2

the amplitude returns but the sign does not.The phase is π (half of the hemisphere's 2π) independently of N, and the surviving amplitude approaches 1 as the steps are refined——measuring intensity alone, an amplitude of 0.987231 comes back at $N = 384$, yet overlaid with a reference it cancels to 0.000041. What has returned is the state, not the phase.

4. A Small Path

Computation 4.1: the phase was obtained for a circle of angular radius 10 degrees near circular polarisation ($\Omega = 0.095456$), traversed in N steps.

N	phase	difference from $-\Omega/2$
6	-0.039621	8.11×10^{-3}
24	-0.047197	5.31×10^{-4}
96	-0.047695	3.33×10^{-5}
384	-0.047726	2.08×10^{-6}
1536	-0.047728	1.30×10^{-7}

Observation 4.2

finer steps converge to the half on any path.Each fourfold increase in the number of projections shrinks the difference by about an order of magnitude, and at $N = 1536$ the phase is within 1.30×10^{-7} of $-\Omega/2 = -0.047728$——the ratio 0.015192 of this small circle's phase to the hemisphere's π equals the ratio 0.015192 of the solid angles.

5. The Separator

Observation 5.1

the separator is whether one measures intensity alone or the phase by interference with a reference.By intensity alone, a finely stepped path returns the light to its original polarisation with almost no loss——overlaid with a reference, half the enclosed solid angle remains as a phase, and for a hemisphere it shows as darkness.

Observation 5.2

the same root as Paper 68.Paper 68 showed that the qubit is a Hopf fibration and that the Berry phase is measurable as the holonomy $-\Omega/2$ of its fibre——the Jones vector $(\cos\frac{\theta}{2}, e^{i\varphi}\sin\frac{\theta}{2})$ used here has the same form as a spin-$1/2$ spinor, and the same function gives the same phase. Since the formula, not merely the number, is the same, this is written as the same root in the sense of Paper 300. The base of the Hopf fibration of Paper 08 is here the Poincaré sphere and the fibre the phase of the polarisation, and the path was followed on the third tool of Paper 207. The 2π that Paper 18 took as the solid angle of a hemisphere is here a phase of π.

Observation 5.3

what is not said.Values of real optical systems, partial polarisation, loss, non-adiabatic effects, and the origin of the phase as a holonomy are not treated——what is shown is that for the polarisation of light too, the phase of a closed path is half the solid angle enclosed, and a hemisphere flips the sign.

6. Census and Closing

1. $\gamma = \arg\langle A|Z\rangle \cdots \langle B|A\rangle$ (Section 1): **the phase of a path of projections.**
2. **Hopf map, 2000 states on the unit sphere** (Computation 2.1): **independent of the overall phase.**
3. $+2.790320$ **for triangle 1** (Computation 2.1): **the core;** $-\Omega/2$**.**
4. $+\Omega/2$ **on the reverse path** (Observation 2.2): **full sphere** $4\pi \leftrightarrow$ **phase** 2π**.**
5. **phase** π **on the hemispherical path** (Computation 3.1): **for** $N = 3$ **to 384.**
6. **amplitude** 0.987231 (Computation 3.1): $N = 384$**, almost fully returned.**
7. **overlaid intensity** 0.000041 (Observation 3.2): **the sign does not return.**
8. -0.047728 **on the small circle** (Computation 4.1): **converging to** $-\Omega/2$**.**
9. **ratio of phases** 0.015192 **= ratio of solid angles** (Observation 4.2): **the half does not depend on the path.**
10. **intensity, or interference** (Observation 5.1): **the separator; Papers 68, 207, 08, 18.**

Closing: **Carrying a polarisation around a closed path shifts its phase by half the solid angle enclosed, and enclosing a hemisphere flips the sign: the same half as the Berry phase of a qubit.The phases of triangular paths made of polariser projections agreed with** $-\Omega/2$ **in all five cases, and on the path that turns a linear polarisation through half a turn and back the phase was** π **independently of** N**——the surviving amplitude returned to** 0.987231 **at** $N = 384$**, yet overlaid with a reference the intensity cancelled to** 0.000041**.On a small circle too the phase converged to** $-\Omega/2$ **as the steps were refined, the reverse path gave** $+\Omega/2$**, and the full sphere** 4π **corresponded to a phase of** 2π**——what returned was the state; the phase remembers half the sky enclosed.Placed among the earlier papers——the same formula as the qubit of Paper 68 (the same root) was counted with light, on the Hopf fibration of Paper 08 and the Poincaré sphere of Paper 207.To be honest——the apparatus values are a model, neither partial polarisation nor loss was included, and the origin of the phase was left to Paper 68.**

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