A Flat Rotation Curve Fixes the Density There and Imposes Only the Enclosed Mass on the Inside

— on a flat stretch the density is $v^2/(4\pi G r^2)$ whatever lies inside, and among power-law densities only $n = 2$, with constant mass per shell, is flat — the separator is whether one asks for the total enclosed mass or for how it grows shell by shell — [Paper 910]

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— From thinking to syncing. —

Abstract

Hearing that a galaxy's rotation speed stays flat far out, it is easy to assume that the flatness fixes the whole arrangement of mass from the centre outward.For spherically symmetric mass distributions, this paper counts numerically what a flat stretch imposes on the density there and on the inside, and what it does not.No new theorem or law is claimed. First, among power-law densities only $n = 2$ is flat——integrating $\rho = r^{-n}$ on a logarithmic grid, the slope of the rotation speed matched $(2 - n)/2$ for five values of n, and only for $n = 2$ did the speed stay the same from radius 1 to 16 with a ratio of mass per shell of 1.000000. The mass enclosed within radius 1 followed the staircase $4\pi/(3 - n)$: 4.188790, 6.283185, 8.377580, 12.566371, 25.132741 (Section 2). **Second, the slope of the speed fixes the density at that place——with $4\pi G r^2 \rho = v^2(1 + 2s)$ (s the logarithmic slope of the speed), the density recovered from the numerical rotation curve of the cored density $1/(1 + r^2)$ differed by at most 3.37×10^{-9} at seven radii. The factor $1 + 2s$ was 2.999880 near the centre and 1.001571 at $r = 1000$, and for the finite-mass e^{-r} it fell to 8.246576×10^{-6} at $r = 20$** (Section 3). **Third, and this is the core. All that a flat stretch imposes on the inside is the enclosed mass——setting the density to $1/(4\pi r^2)$ outside radius 1 and filling the inside with r^{-m} ($m = 0$, 1, 1.5, 2) so that the enclosed mass matches, the outside speed stayed 1.000000 in every case and the density merely jumped by a factor $3 - m$ at the join. Increasing the inner mass by 10 per cent raised the speed at radius 1.5 to 1.032796, and it kept falling as $\sqrt{1 + 0.1/r}$** (Section 4). **Fourth, an isothermal sphere, starting from a core, still moves toward r^{-2}——integrating the isothermal sphere equation in $\ln \xi$, v^2/σ^2 reached 1.998748 at $\xi = 10^6$. The first maximum was 2.517551 at $\xi = 8.9931$, a speed 1.121952 times the distant one, and the spacing of maxima in $\ln \xi$ was 4.749797 (linearisation: $4\pi/\sqrt{7} = 4.749642$)** (Section 5). **Fifth, with rounded model values, a flat 200 km/s imposes 9.300341×10^{10} solar masses inside a radius of 10 kpc——the density at that radius is 7.400976×10^{-3} solar masses per pc^3 (0.299297 hydrogen atoms per cm^3), and the mass per shell is 9.300341×10^9 solar masses per kpc** (Section 6). **Sixth, the separator is whether one asks for the total enclosed mass or for how it grows shell by shell ——Keplerian fall-off asks only for the total, whereas a flat stretch fixes the growth per shell at v^2/G. Only the total is left for the inside, and its arrangement is not asked for** (Section 7). **Keywords**: rotation curve, singular isothermal sphere, enclosed mass, isothermal sphere, density profile, spherical symmetry.

Scope of this paper (scope note): No new theorem or law is claimed——the observation of flat rotation curves (Rubin and Ford 1970), the enclosed mass $M = v^2 r/G$ and density $v^2/(4\pi G r^2)$, the $v^2 = 2\sigma^2$ of the singular isothermal sphere, and the isothermal sphere equation with its solutions approaching the singular solution while oscillating (Chandrasekhar 1939) are all known (Binney and Tremaine 2008).Neither the existence nor the nature of dark matter is discussed——only the density and mass that the flatness requires are counted; what that mass is made of is not asked.No change to the law of gravity is entered——the counting stays with Newtonian gravity and spherical symmetry.Discs are not treated——mass is placed spherically, and the mass a flattened distribution needs for the same speed is not counted.No values of real galaxies are claimed——200 km/s and 10 kpc are rounded model values, and where a flat stretch ends is not entered.Relation to earlier papers: Paper 01 wrote in its scope note that $\rho \propto r^{-2}$ is known in astrophysics as the singular isothermal sphere——this paper is set beside that scope note (no earlier occupant). Here it is counted that this r^{-2} is the density imposed on a flat stretch, and also where an isothermal sphere starting from a core ends up.Paper 337 obtained the enclosed mass $M = v^2 R/G$ from the rotation speed and computed a Hill radius——here the same formula is read stretch by stretch to recover the density.Paper 51 withdrew exponential screening of gravity, giving as one reason its conflict with observations such as galactic rotation curves ——here, with unscreened gravity, what flatness imposes on the density is counted.Paper 782 took a flat rotation speed as a given value and counted how stars pass through the pattern of the arms——here what that flatness itself imposes on the mass is counted.Paper 141 integrated the Lane--Emden equation for $n = 3$ itself to obtain its constants——here the isothermal case is integrated as a two-variable system.Paper 907 derived the expansion equation using the fact that matter outside a shell does not affect the shell's motion——here the same fact is used for circular motion, and it is counted that the arrangement inside does not affect the outside speed.What is added is confirming by numerical integration that among power-law densities only $n = 2$ is flat, recovering the density with the factor $1 + 2s$ of the local relation, showing that filling the inside with another power leaves the outside flat and makes the density jump by a factor $3 - m$ at the join, counting the overshoot and logarithmic period of the isothermal sphere, and placing the separator on the total versus the growth per shell.

1. Notation and the Question

Setting: for a spherically symmetric mass distribution, let $M(r)$ be the mass inside radius r and $v(r)$ the circular speed. Since mass outside a shell does not affect circular motion, the following hold.

$$v^2 = \frac{GM(r)}{r}, \qquad \frac{dM}{dr} = 4\pi r^2 \rho, \qquad 4\pi G r^2 \rho = \frac{d(rv^2)}{dr} = v^2(1 + 2s), \qquad s = \frac{d\ln v}{d\ln r}$$

On a flat stretch ($s = 0$), $\rho = v^2/(4\pi G r^2)$; for Keplerian fall-off ($s = -1/2$), $\rho = 0$; for rigid rotation ($s = 1$), $\rho = 3v^2/(4\pi G r^2)$.For the isothermal sphere, with velocity dispersion σ, central density ρ_0, $r_s^2 = \sigma^2/(4\pi G \rho_0)$, $\xi = r/r_s$ and $\rho = \rho_0 e^{-\psi}$, the following equation is solved.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right)=e^{-\psi},\qquad z=\xi\frac{d\psi}{d\xi}=\frac{v^2}{\sigma^2},\qquad w=\xi^2e^{-\psi}=\frac{2\rho r^2}{\sigma^2/(2\pi G)}$$

With $t=\ln\xi$ this becomes the two-variable system $dz/dt=w-z$, $dw/dt=w(2-z)$, whose fixed point $(2,2)$ is the singular isothermal sphere. The eigenvalues of the linearisation there are $(-1\pm i\sqrt{7})/2$, so it oscillates with period $4\pi/\sqrt{7}$ in $\ln\xi$ and the amplitude decays as $\xi^{-1/2}$.Mass was integrated by the trapezoid rule on a grid evenly spaced in $\ln r$, with separate grids inside and outside wherever the density jumps.

Question: what does a flat stretch of rotation speed impose on the density and on the inner mass, and what does it not?

2. Power-Law Densities and Rotation Speed

Computation 2.1: $\rho=r^{-n}$ ($G=1$) was integrated, and from radius 1 to 16 the slope of the speed and the ratio of mass per shell (per unit radius from 8 to 16, divided by that from 1 to 2) were obtained.

n	mass enclosed within radius 1	slope of speed	$v(16)/v(1)$	ratio of mass per shell
0	4.188790	1.000000	16.000000	64.000000
1	6.283185	0.500000	4.000000	8.000000
1.5	8.377580	0.250000	2.000000	2.828427
2	12.566371	0.000000	1.000000	1.000000
2.5	25.132741	−0.250000	0.500000	0.353553

> **Observation 2.2**
>
> among power-law densities only $n=2$ is flat.The slope of the speed matched $(2-n)/2$ within 10^{-6} for five values of n, and only for $n=2$ did the speed stay the same with a ratio of mass per shell of 1——the mass enclosed within radius 1 matched $4\pi/(3-n)$; for $n=2$ it is 4π, with speed $\sqrt{4\pi}=3.544908$.

3. Recovering the Density from the Slope of the Speed

Computation 3.1: the cored density $\rho=1/(1+r^2)$ was integrated to make a rotation curve, s was differentiated numerically, and $v^2(1+2s)/(4\pi Gr^2)$ was compared with the original density.

r	s	$1+2s$	relative difference of recovered density
0.01	0.999940	2.999880	3.37×10^{-9}
0.1	0.994049	2.988098	3.30×10^{-9}
1	0.664946	2.329892	1.03×10^{-9}
3	0.271009	1.542018	6.17×10^{-10}
10	0.080441	1.160881	4.83×10^{-10}
100	0.007877	1.015754	3.87×10^{-10}
1000	0.000786	1.001571	3.74×10^{-10}

Observation 3.2

the slope of the speed fixes the density at that place.The factor $1+2s$ approaches the 3 of rigid rotation near the centre and the 1 of a flat curve far out, and the speed matched the closed form $v/v_\infty=\sqrt{1-\arctan r/r}$ (0.923518 at $r=10$, 0.999215 at $r=1000$)——for the finite-mass e^{-r}, $1+2s$ falls toward 0: 4.810638×10^{-1} at $r=5$ and 8.246576×10^{-6} at $r=20$. The slope of the speed is -0.499333 from $r=10$ to 80 (the closed form also gives -0.499333) and -0.500000 from $r=40$ to 160; at radius 10 the enclosed mass is still growing.

4. What a Flat Stretch Imposes on the Inside

Computation 4.1: with $G=1$ and $v=1$, the density outside radius 1 was set to $\rho=1/(4\pi r^2)$ and the inside filled with $\rho=kr^{-m}$. Matching an enclosed mass of 1 gives $k=(3-m)/(4\pi)$.

inner m	coefficient k	density jump at the join (inside/outside)	$v(0.5)$	$v(1.5), v(3), v(10), v(100)$
0	0.238732	3.000000	0.500000	1.000000
1	0.159155	2.000000	0.707107	1.000000
1.5	0.119366	1.500000	0.840896	1.000000
2	0.079577	1.000000	1.000000	1.000000
uniform inside, mass up 10 per cent				1.032796, 1.016530, 1.004988, 1.000500

Observation 4.2

this is the core. All that a flat stretch imposes on the inside is the enclosed mass.Whatever power fills the inside, as long as the enclosed mass matches $v^2 r_1/G$ the outside speed stayed 1.000000 at four radii, and the density merely jumped by a factor $3-m$ at the join. It joins without a jump only when the inside also has $m=2$——increasing the inner mass by 10 per cent, the outside speed matched $\sqrt{1+0.1/r}$, 1.032796 at radius 1.5 and 1.000500 at radius 100, and kept falling instead of being flat. A flat stretch fixes the density there at $v^2/(4\pi G r^2)$ and imposes a single total on the inside.

5. The Isothermal Sphere

Computation 5.1: **starting from a series at $\xi = 10^{-3}$, the system was integrated in $t=\ln\xi$ by the Runge--Kutta method with step 10^{-3} up to $\xi = 10^{6}$.**

ξ	v^2/σ^2	$\rho r^2/(\sigma^2/2\pi G)$
1	0.302901	0.426572
3	1.551346	1.553860
10	2.510611	1.191798
30	2.083093	0.839614
100	1.842785	0.924163
10^3	2.047800	1.026655
10^4	1.985619	0.990856
10^5	2.004272	1.003109
10^6	1.998748	0.998960

Observation 5.2

an isothermal sphere, starting from a core, still moves toward r^{-2}.The maxima of v^2/σ^2 are 2.517551 at $\xi = 8.9931$, 2.047949 at $\xi = 1056.0987$ and 2.004461 at $\xi = 122043.6136$; the speed at the first maximum is 1.121952 times the distant speed $\sqrt{2}\sigma$——the spacings of maxima in $\ln\xi$ were 4.765877 and 4.749797, approaching the linearisation's $4\pi/\sqrt{7} = 4.749642$ (a factor of 115.542872 in ξ each time), and the decay rate of the excess over 2 was 0.499166 and 0.499992, approaching $1/2$. The rotation curve of a cored isothermal sphere overshoots once and then flattens while swinging.

6. Rounded Model Values

Computation 6.1: a flat speed of 200 km/s and a radius of 10 kpc (rounded model values) were taken, with $G = 6.67430 \times 10^{-11}$ and a solar mass of 1.98841×10^{30} kg.

quantity	value
enclosed mass $v^2 r/G$	9.300341×10^{10} solar masses
density at that radius $v^2/(4\pi G r^2)$	7.400976×10^{-3} solar masses per pc^3
the same density in hydrogen atoms	0.299297 per cm^3
mass per shell v^2/G	9.300341×10^{9} solar masses per kpc
$\sigma = v/\sqrt{2}$ of the singular isothermal sphere	141.421356 km/s
time for one revolution	307.182486 million years
enclosed mass if flat out to 20 kpc	1.860068×10^{11} solar masses
density at 20 kpc	1.850244×10^{-3} solar masses per pc^3

Observation 6.2

as long as a flat stretch continues, the enclosed mass grows in proportion to the radius and the density thins as the square of the radius.Going from 10 kpc to 20 kpc, the enclosed mass doubles to 1.860068×10^{11} solar masses and the density falls to a quarter, 1.850244×10^{-3} solar masses per pc^3——both numbers are fixed by the flat speed and the radius alone, and do not depend on how the mass is arranged inside.

7. The Separator

Observation 7.1

the separator is whether one asks for the total enclosed mass or for how it grows shell by shell.A stretch of Keplerian fall-off asks only for the inner total, and the density there is 0——a flat stretch fixes the growth per shell at v^2/G and the density there at $v^2/(4\pi G r^2)$. What is left for the inside is only the total $v^2 r_1/G$, and its arrangement does not show in the outside speed.

Observation 7.2

placed among the earlier papers.The r^{-2} of the singular isothermal sphere, named in the scope note of Paper 01, turned out to be the density imposed on a flat stretch and also where an isothermal sphere starting from a core ends up——the $M = v^2 R/G$ of Paper 337 was read stretch by stretch to recover the density, counting with gravity free of the screening that Paper 51 withdrew. What the flatness taken as given in Paper 782 imposes on mass was counted, the Lane--Emden integration of Paper 141 was carried over to the isothermal case, and the fact from Paper 907 that the outside of a shell has no effect was used for circular motion.

Observation 7.3

what is not said.The existence and nature of dark matter, changes to the law of gravity, the shape of discs, the rotation speeds and radii of real galaxies, and where flat stretches end are not treated——what is shown is that, for spherically symmetric Newtonian gravity, a flat stretch fixes the density there and the inner total, and does not fix the arrangement inside.

8. Census and Closing

1. $4\pi G r^2 \rho = v^2(1 + 2s)$ (Section 1): **the local relation.**
2. **flat only for $n = 2$** (Computation 2.1): **ratio of mass per shell 1.000000.**
3. **enclosed mass $4\pi/(3 - n)$** (Computation 2.1): **12.566371 for $n = 2$.**
4. **recovered density within 3.37×10^{-9}** (Computation 3.1): **seven radii.**
5. **$1 + 2s$ from 2.999880 to 1.001571** (Computation 3.1): **from rigid rotation to flat.**
6. **density jump $3 - m$ at the join** (Computation 4.1): **outside speed stays 1.000000.**
7. **1.032796 with 10 per cent more mass** (Computation 4.1): **not flat; the core.**
8. **first maximum 1.121952 times** (Observation 5.2): **the overshoot of the isothermal sphere.**
9. **spacing 4.749797** (Observation 5.2): **$4\pi/\sqrt{7} = 4.749642$.**
10. **total, or growth per shell** (Observation 7.1): **the separator; Papers 01, 337, 51, 782, 141, 907.**

Closing: **Where a rotation curve is flat, the density is fixed at $v^2/(4\pi G r^2)$ whatever lies inside, and all that is imposed on the inside is the enclosed mass.Among power-law densities only $n = 2$, with constant mass per shell, is flat, and the local relation $4\pi G r^2 \rho = v^2(1 + 2s)$ recovered the density from a numerical rotation curve——filling the inside with another power, the outside stayed flat as long as the enclosed mass matched, and the density merely**

jumped by a factor $3 - m$ at the join. With 10 per cent more inner mass the speed rose to 1.032796 and kept falling.A cored isothermal sphere overshot to 1.121952 times the distant speed and then approached the r^{-2} of $v^2 = 2\sigma^2$, swinging with spacing 4.749797 in $\ln \xi$——flatness fixes the growth per shell and leaves only the total for the inside.Placed among the earlier papers——the singular isothermal sphere named in the scope note of Paper 01 was counted as the density of a flat stretch and as where an isothermal sphere ends up, and the enclosed mass of Paper 337 was read stretch by stretch.To be honest——this is a computation in spherically symmetric Newtonian gravity, and does not enter the existence and nature of dark matter, the shape of discs, or the values of real galaxies.

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