

Taking m as the Area of the Unit Sphere Makes the Coefficient of Temperature the Volume of the Unit Ball in Every Dimension

— for every d from 1 to 8, the coefficient of temperature matches the volume of the unit ball, and what it is is the integral over the sphere of a squared direction component — the separator is whether the average is over all directions or over the transverse ones — [Paper 912]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

September 2026

— *From thinking to syncing.* —

Abstract

Paper 20 wrote temperature as $kT = (4\pi/3)v^2$ from $m = 4\pi$, and left the fact that the coefficient $4\pi/3$ is the volume of the unit ball as something that “appears as a coefficient”. This paper moves the dimension d of space, takes m as the area S_{d-1} of the unit sphere in d dimensions, counts whether the agreement is confined to $d = 3$ or holds in every dimension, and, if it holds, shows what it is. No new theorem or law is claimed. First, the coefficient of temperature matches the volume of the unit ball in every dimension—counting the coefficient of temperature (kT divided by the mean square speed) by sampling the Maxwell distribution gives **4.187827** against **4.188790** at $d = 3$ and **5.264092** against **5.263789** at $d = 5$, and every difference from $d = 1$ to 8 is within 1.37 times the standard error measured over 20 batches (Section 2). Second, and this is the core, what it is is the integral over the sphere of a squared direction component—by the divergence theorem this surface integral equals the volume of the unit ball. A grid integration at $d = 3$ gives **4.188790205** on the surface side, agreeing with the volume side to a difference of 1.6×10^{-10} , and sampling gives agreement within 1.07 standard errors for $d = 2$ to 8 (Section 3). Third, the coefficient of pressure and the coefficient of the energy of a rotating shell give the same volume—at $d = 3$ the samples are **4.189485** and **4.189691**, the moment of inertia of a shell with $m = 4\pi$ is **8.377580**, and half of it, **4.188790**, is the coefficient of the rotational energy (Section 4). Fourth, with m fixed at 4π , the agreement holds only at $d = 3$ —the coefficient $4\pi/d$ is **3.141593** at $d = 4$, while the volume of the unit ball is **4.934802**. The agreement holds in every dimension because m was chosen as the area of the unit sphere (Section 5). Fifth, quantities that take the transverse average come out as $d - 1$ times the volume of the unit ball—**8.377580** at $d = 3$ and **14.804407** at $d = 4$, agreeing with the volume only at $d = 2$. The $2/3$ of Larmor’s formula and the $4/3$ of the electromagnetic mass belong to this family (Section 6). Sixth, the separator is whether the average is over all directions or over the transverse ones—averaging over all directions gives the volume of the unit ball in every dimension, largest at $d = 5$ with **5.263789**; averaging transversely gives $d - 1$ times it, and at $d = 3$ the $4\pi/3$ and the $8\pi/3$ are different numbers of the two families (Section 7). **Keywords:** equipartition, volume of the unit ball, divergence theorem, Maxwell distribution, space dimension, temperature.

Scope of this paper (scope note): No new theorem or law is claimed—the equipartition of energy (Maxwell 1860), the volume of the unit ball in d dimensions being $1/d$ of its area, the divergence theorem, and the spherical average $1/d$ of a squared direction component are all standard. Temperature is not said to be a volume—temperature has the dimension of speed squared, and the withdrawal in the second edition of Paper 20 stands. Only the coefficient is at issue. Mass is not said to be an area or a solid angle—taking m equal in number to the area of the unit sphere is a choice of units, and, as in the second edition of Paper 60, the chosen 4π and the 4π of solid angle are different things. No reality is claimed for gases in higher dimensions—a gas with $d \neq 3$ is a way of counting how the coefficient depends on d . No quantum or relativistic corrections are entered—only the equipartition of a classical ideal gas is treated. Relation to earlier papers: Paper 20 derived $kT = (4\pi/3)v^2$ from $m = 4\pi$ and equipartition, and kept the coefficient's being the volume of the unit ball at the level of a coefficient—here that agreement is counted as holding in every dimension, and shown to be an integral of a squared direction over the sphere. Paper 19 wrote the $4\pi/3$ of $P = (4\pi/3)nv^2$ as the volume of the three-dimensional unit ball—here the coefficient of pressure is counted as the same integral, a volume in every dimension. Paper 18 wrote kinetic energy as $2\pi v^2$ with $m = 4\pi$ —here that coefficient times $2/d$ gives the coefficient of temperature, which becomes the volume of the unit ball when m is taken as the area of the unit sphere. Paper 59 showed that the volume of the unit ball is greatest in five dimensions, and that the 5 follows from setting the radius to 1—here choosing m as the area of the unit sphere makes the coefficient of temperature greatest in five dimensions too. Paper 905 counted the $2/3$ of Larmor's formula as the directional average $1 - 1/d$ —here that is placed in the transverse family and separated from the family of all directions. Paper 907 obtained the $8\pi/3$ of the critical density from the $1/2$ of kinetic energy and the $4\pi/3$ of a ball's volume, and showed that the transverse integral that looks like the same $8\pi/3$ splits into a different number at $d = 4$ —here the transverse integral is used as $d - 1$ times the volume of the unit ball. Paper 908 showed that the $4/3$ of the electromagnetic mass comes not from a ball's volume but from averaging over field directions—here the $4\pi/3$ of temperature and pressure is shown, the other way round, to be the volume of the ball itself in every dimension. Paper 902 read the 4π of black-body radiation as the area of the unit sphere in k space—here m is chosen equal in number to the area of the unit sphere, and the integral over that surface is counted. What is added: counting by sampling that, with m as the area of the unit sphere, the coefficient of temperature is the volume of the unit ball for $d = 1$ to 8, showing what it is as the integral of a squared direction over the sphere, on both sides of the divergence theorem, counting that pressure and a rotating shell belong to the same family, showing that with m fixed at 4π the agreement holds only at $d = 3$, and placing the separator on “the average over all directions, or the transverse average”.

1. Notation and the Question

Setting: write S_{d-1} for the area and V_d for the volume of the unit ball in d dimensions.

$$S_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \quad V_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} = \frac{S_{d-1}}{d}$$

At $d = 3$, $S_2 = 4\pi$ and $V_3 = 4\pi/3$; at $d = 1$, $S_0 = 2$ (two points) and $V_1 = 2$ (a segment of length 2); at $d = 2$, $V_2 = \pi$. In an ideal gas in d dimensions equipartition gives $\frac{1}{2}kT$ per direction, so:

$$\left\langle \frac{1}{2}mv^2 \right\rangle = \frac{d}{2}kT \implies \frac{kT}{\langle v^2 \rangle} = \frac{m}{d} \xrightarrow{m=S_{d-1}} \frac{kT}{\langle v^2 \rangle} = \frac{S_{d-1}}{d} = V_d$$

At $d = 3$ this is the $kT = (4\pi/3)v^2$ of Paper 20 (with v^2 the mean square). The check draws 10^5 velocities 20 times from a normal distribution of variance kT/m per component (the Maxwell distribution), counts $kT/\langle v^2 \rangle$, and measures the standard error from the spread of the 20 batches.

Question: with m taken as the area of the unit sphere, is the coefficient of temperature the volume of the unit ball only at $d = 3$, or in every dimension? If in every dimension, what is it?

2. The Volume of the Unit Ball in Every Dimension

Computation 2.1: $kT/\langle v^2 \rangle$ with $m = S_{d-1}$ (sampled) and V_d . z is the difference divided by the standard error.

d	S_{d-1}	V_d	$kT/\langle v^2 \rangle$ (sampled)	z
1	2.000000	2.000000	1.999826	−0.12
2	6.283185	3.141593	3.142312	+0.31
3	12.566371	4.188790	4.187827	−0.40
4	19.739209	4.934802	4.935205	+0.14
5	26.318945	5.263789	5.264092	+0.13
6	31.006277	5.167713	5.164820	−1.36
7	33.073362	4.724766	4.724542	−0.17
8	32.469697	4.058712	4.060158	+1.37

Observation 2.2

the coefficient of temperature matches the volume of the unit ball for every d from 1 to 8. Every difference is within 1.37 standard errors, and S_{d-1}/d and V_d differ by less than 10^{-12} in all eight dimensions—the $4\pi/3$ of Paper 20 is not a three-dimensional accident.

3. What It Is: the Integral of a Squared Direction Component over the Sphere

Setting: let n be the outward unit normal of the unit sphere. On the unit sphere $n = x$, so applying the divergence theorem to the field $F = (x_1, 0, \dots, 0)$ gives:

$$\oint_{S_{d-1}} n_1^2 dS = \oint_{S_{d-1}} x_1 n_1 dS = \int_{B_d} \frac{\partial x_1}{\partial x_1} dV = V_d$$

For velocities with no preferred direction, speed and direction are independent, so $\langle v_1^2 \rangle = \langle v^2 \rangle \langle n_1^2 \rangle$, and equipartition $kT = m \langle v_1^2 \rangle$ gives $kT / \langle v^2 \rangle = m \langle n_1^2 \rangle$. With $m = S_{d-1}$, area times average is the surface integral itself:

$$\frac{kT}{\langle v^2 \rangle} = S_{d-1} \langle n_1^2 \rangle = \oint_{S_{d-1}} n_1^2 dS = V_d$$

Computation 3.1: the surface integral $S_{d-1} \langle n_1^2 \rangle$ (sampled), and the volume of the unit ball counted by throwing points into a cube (sampled).

d	surface integral (sampled)	z	ball volume (sampled)	z	V_d
2	3.140273	−0.92	3.140192	−1.20	3.141593
3	4.191567	+1.07	4.189908	+0.58	4.188790
4	4.939248	+0.99	4.937384	+0.35	4.934802
5	5.264264	+0.13	5.253248	−1.52	5.263789
6	5.168169	+0.12	5.176256	+0.61	5.167713
7	4.725843	+0.25	4.718848	−0.33	4.724766
8	4.059469	+0.23	4.087552	+1.31	4.058712

Grid integrations independent of sampling give $\oint \cos^2 \varphi d\varphi = 3.141592654$ at $d = 2$ ($V_2 = \pi$) and $\oint \cos^2 \theta d\Omega = 4.188790205$ at $d = 3$ ($V_3 = 4\pi/3$), differing by 1.6×10^{-10} from the 4.188790205 of the volume side accumulated on a radial grid at $d = 3$.

Observation 3.2

choosing m as the area of the unit sphere turns “mass times a directional average” into “an integral over the sphere”, and the divergence theorem turns that into a volume. The $1/d$ of equipartition is the average of a squared direction, the $1/d$ of the unit ball’s volume is its ratio to the area, and the two stand on the two sides of the same equality—the sampled surface integral matched V_d within 1.07 standard errors and the sampled ball volume within 1.52.

4. The Same Family: Pressure and a Rotating Shell

Setting: the pressure on a wall is $P = nm\langle v_1^2 \rangle$, so the coefficient $P/(n\langle v^2 \rangle) = m\langle n_1^2 \rangle$ is the same integral as for temperature. A thin shell of radius r turning in one plane at angular speed ω has moment of inertia $I = mr^2\langle n_1^2 + n_2^2 \rangle$, and its rotational energy is:

$$\frac{\frac{1}{2}I\omega^2}{r^2\omega^2} = \frac{1}{2} m \langle n_1^2 + n_2^2 \rangle = m\langle n_1^2 \rangle \xrightarrow{m=S_{d-1}} V_d$$

Computation 4.1: the coefficient of pressure and the coefficient of a rotating shell with $m = S_{d-1}$ (both sampled).

d	coefficient of pressure	z	coefficient of rotating shell	z	V_d
2	3.143941	+1.00	3.141593	+0.00	3.141593
3	4.189485	+0.20	4.189691	+0.90	4.188790
4	4.930971	−0.81	4.931856	−1.29	4.934802
5	5.270911	+1.41	5.267303	+1.33	5.263789
6	5.175761	+1.42	5.174199	+2.81	5.167713
7	4.725018	+0.06	4.726354	+0.62	4.724766
8	4.056544	−0.51	4.059057	+0.14	4.058712

Observation 4.2

the coefficients of pressure and of a rotating shell also match the volume of the unit ball. At $d = 2$ the plane of rotation is every direction, so the shell's value has no spread and is exactly 3.141593—at $d = 3$ the moment of inertia of a shell with $m = 4\pi$ is **8.377580**, and half of it, 4.188790, is the coefficient of rotational energy. At $d = 4$, counting the two components in the plane of rotation gives $I = 9.869604$, while counting the components perpendicular to an axis gives 14.804407; only in three dimensions are both the same two components, so they cannot be told apart.

5. With m Fixed at 4π

Computation 5.1: the coefficient $4\pi/d$ with $m = 4\pi$ fixed in every dimension, and V_d .

d	$4\pi/d$	V_d
1	12.566371	2.000000
2	6.283185	3.141593
3	4.188790	4.188790
4	3.141593	4.934802
5	2.513274	5.263789
6	2.094395	5.167713
7	1.795196	4.724766
8	1.570796	4.058712

Observation 5.2

with m fixed at 4π , the coefficient agrees with the volume of the unit ball only at $d = 3$. It agrees in every dimension because m was chosen as the area of the unit sphere in that dimension—just as Paper 59 traced the maximum at 5 of the unit ball's volume to “setting the radius to 1”, here “setting m to the area of the unit sphere” is the condition for the agreement.

6. The Transverse Average

Setting: the sum of the squared components perpendicular to one direction is $1 - n_1^2$, and its integral over the sphere is the area minus the volume:

$$\oint_{S_{d-1}} (1 - n_1^2) dS = S_{d-1} - V_d = (d - 1) V_d$$

Computation 6.1: the transverse integral and its ratio to V_d .

d	$S_{d-1} - V_d$	ratio to V_d
1	0.000000	0.000000
2	3.141593	1.000000
3	8.377580	2.000000
4	14.804407	3.000000
5	21.055156	4.000000
6	25.838564	5.000000
7	28.348596	6.000000
8	28.410985	7.000000

The grid integration of $\oint \sin^2 \theta d\Omega$ at $d = 3$ gives 8.377580410, matching $8\pi/3$ within 10^{-9} .

Observation 6.2

the transverse integral is $d - 1$ times the volume of the unit ball, and agrees with it only at $d = 2$. The $2/3$ of Larmor’s formula in Paper 905 is $\langle \sin^2 \theta \rangle = 1 - 1/d$, and the $4/3$ of the electromagnetic mass in Paper 908 is twice that; both belong to this family—as Paper 907 showed, a number that looks like the same $8\pi/3$ at $d = 3$ splits into 9.869604 and 14.804407 at $d = 4$. The $4\pi/3$ of temperature and pressure lives in the other family.

7. The Separator

Observation 7.1

the separator is whether the average is over all directions or over the transverse ones. With the average over all directions ($1/d$), the coefficient with m as the area of the unit sphere is the volume of the unit ball in every dimension—with the transverse average ($1 - 1/d$) it is $d - 1$ times that, and agrees with the volume only at $d = 2$.

Observation 7.2

the dimension where the coefficient is largest. From $d = 1$ to 20 the volume of the unit ball is largest at $d = 5$, **5.263789** ($V_4 = 4.934802$, $V_6 = 5.167713$), so the coefficient of temperature with m as the area of the unit sphere is also largest in five dimensions—as in Paper 59, this 5 follows from how the units were set, and is not a physically special dimension.

Observation 7.3

what is not said. That temperature is a volume, that mass is an area or a solid angle, that gases in higher dimensions exist, and quantum or relativistic corrections are not treated—what is shown is that, with m taken equal in number to the area of the unit sphere, the coefficients of temperature, pressure, and a rotating shell are, as the integral of a squared direction component over the sphere, the volume of the unit ball in every dimension.

8. Roll Call and Closing

1. $kT/\langle v^2 \rangle = m/d$ (Section 1): V_d with $m = S_{d-1}$.
2. **4.187827** against **4.188790** at $d = 3$ (Computation 2.1): within **1.37** standard errors in eight dimensions.
3. surface side **4.188790205** (Section 3): both sides of the divergence theorem; the core.
4. moment of inertia of the shell **8.377580** (Observation 4.2): half of it is **4.188790**.
5. **3.141593** at $d = 4$ with $m = 4\pi$ fixed (Computation 5.1): agreement only at $d = 3$.
6. transverse **14.804407** at $d = 4$ (Computation 6.1): three times the volume.
7. largest at $d = 5$, **5.263789** (Observation 7.2): Paper 59.

8. all directions, or transverse (Observation 7.1): the separator; Papers 18, 19, 20, 59, 60, 902, 905, 907, 908.

Closing: Taking m as the area of the unit sphere makes the coefficient of temperature the volume of the unit ball in every dimension. The coefficient of temperature counted by sampling the Maxwell distribution matched the volume of the unit ball within 1.37 standard errors for every d from 1 to 8, and the $4\pi/3$ of Paper 20 was not a three-dimensional accident——what it is is the integral over the sphere of a squared direction component, which the divergence theorem turns into a volume. At $d = 3$ a grid integration gave 4.188790205 on the surface side, agreeing with the volume side to a difference of 1.6×10^{-10} . The coefficients of pressure and of a rotating shell belong to the same family, and with m fixed at 4π the agreement holds only at $d = 3$. Quantities that take the transverse average are $d - 1$ times the volume, and the $2/3$ of Larmor's formula and the $4/3$ of the electromagnetic mass live there——the separator is the average over all directions, or the transverse average. Placed among the earlier papers——the agreement that Paper 20 left at the level of a coefficient is derived from a surface integral as an equality holding in every dimension, and set beside the “ $4/3$ that is not a ball's volume” of Paper 908, dividing the two families. To be honest——temperature is not said to be a volume, nor mass an area or a solid angle. With m chosen equal in number to the area of the unit sphere, the coefficients look like this, and no further.

References

1. Yamagishi, Y. (2026). *The Yamagishi Overclock Temperature Equation: Derivation of $kT = \frac{4}{3}\pi v^2$ and Consistency with the Standard Equation of State* [Paper 20]. Zenodo. 10.5281/zenodo.21601035
2. Yamagishi, Y. (2026). *The Yamagishi Spherical Compression Pressure Equation* [Paper 19]. Zenodo. 10.5281/zenodo.21600912
3. Yamagishi, Y. (2026). *The Yamagishi Kinetic Energy Equation* [Paper 18]. Zenodo. 10.5281/zenodo.21430706
4. Yamagishi, Y. (2026). *The Volume of the Unit Ball Is Greatest in Five Dimensions* [Paper 59]. Zenodo. 10.5281/zenodo.21597865
5. Yamagishi, Y. (2026). *c as the Ball of Velocity Space* [Paper 60]. Zenodo. 10.5281/zenodo.21598012
6. Yamagishi, Y. (2026). *The Same c^3 Counts Differently: Black-Body Radiation's 3 Counts Dimensions, the Planck Length's Does Not* [Paper 902]. Zenodo. 10.5281/zenodo.22800314
7. Yamagishi, Y. (2026). *The 3 in Larmor's $2/3$ and the 3 in Its c^3 Are Both the Space Dimension* [Paper 905]. Zenodo. 10.5281/zenodo.22803759
8. Yamagishi, Y. (2026). *Cosmic Expansion Follows Exactly from a Newtonian Shell for Dust, but for Light the Deceleration Is Twice the Shell's* [Paper 907]. Zenodo. 10.5281/zenodo.22804787
9. Yamagishi, Y. (2026). *The $4/3$ of the Electromagnetic Mass Comes Not from a Ball's Volume but from Averaging over Field Directions* [Paper 908]. Zenodo. 10.5281/zenodo.22804819
10. Maxwell, J. C. (1860). Illustrations of the dynamical theory of gases. *Philosophical Magazine*.
11. Huang, K. (1987). *Statistical Mechanics*. Wiley.

— *From thinking to syncing.* —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)