

Antenna Gain Averages to 1 over All Directions, and the Factor That Turns Beamwidth into Gain Depends on the Power in the Skirts

— gain adds no power and measures how the 4π of solid angle is shared out, and at the same half-power beamwidth the gain is smaller the more power escapes the main beam — the separator is the fraction of power outside the main beam — [Paper 918]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

September 2026

YOI: <https://www.yamagishiyuuki.com/yoi/918>

— *From thinking to syncing.* —

Abstract

Antenna gain (here the directivity, without loss) states how many times the power sent in one direction exceeds what it would be if spread evenly over all directions. This paper confirms what gain averages to over all directions, and counts how much the factor that estimates gain from the half-power beamwidth changes with the shape of the pattern. No new theorem or law is claimed. First, gain averages to 1 over all directions: for a short dipole, a half-wave dipole, four cosine-power patterns and two circular apertures alike, averaging gain over the 4π of solid angle gave 1, within 4.6×10^{-15} . The short dipole has 1.500000 and the half-wave dipole **1.640922** (Section 2). Second, the factor that turns the half-power beamwidth into gain changes with the pattern: gain times the square of the half-power beamwidth (in square degrees) fell from 48600.0 for cosine to the power 2 to 39131.5 for the power 10, 36686.7 for 100 and 36435.5 for 1000 (Section 2). Third, shapes without skirts come down to the Gaussian limit: the Gaussian limit is **36407.5** square degrees, 0.882542 of the 41252.961 square degrees in 4π steradians (Section 2). Fourth, a circular aperture, which has skirts, has a factor below the Gaussian limit: a uniform circular aperture gives 34841.9 at $ka = 10$ and **34392.7** at $ka = 30$. A circular aperture sends **0.162215** of its power outside the first dark ring (Section 3). Fifth, the separator is the fraction of power outside the main beam: gain is the share of 4π given to the main beam, so at the same half-power beamwidth the gain is smaller the more power leaves the main beam. The estimate of 41253 divided by the square of the beamwidth was off by factors from 1.178097 to 0.833703 across the patterns (Section 4). **Keywords:** antenna gain, directivity, half-power beamwidth, solid angle, sidelobes, circular aperture.

Scope of this paper (scope note): No new theorem or law is claimed: the definition of directivity, the patterns and directivities of dipoles, the diffraction pattern of a uniform circular aperture, and the rules of thumb estimating directivity from half-power beamwidth are all standard. No antenna is designed: feeding, matching and values for real products are not treated. No loss is included: gain is taken equal to directivity, without reduction by efficiency. Which rule of thumb is right is not judged: whether 41253 or another factor should be used is not stated; the values for each pattern are only set side by side. Only rotationally symmetric patterns are used: beams of different widths in two planes are not treated. The circular aperture and the cosine-power patterns are taken to radiate into the forward hemisphere only. Relation to earlier papers: Paper 384, in its scope note, did not enter the theory of antenna radiation resistance or directivity; this paper sits in that seat of directivity (no earlier occupant). Paper 335 showed that for radar the round trip changes the power of distance. Here, the gain that enters that equation is counted as the ratio that shares out the 4π of solid angle so that its average over all directions is 1. What is added: confirming for eight patterns that gain averages to 1 over all directions, setting down the factor that turns half-power beamwidth into gain, showing that the factor for the cosine-power patterns comes down to the Gaussian limit of 36407.5 square degrees, finding that the factor for a circular aperture, which has skirts, is below that limit, and placing the separator on “the fraction of power outside the main beam”.

1. Notation and the Question

Setting: let U be the radiation intensity in direction (θ, ϕ) . The directivity D (gain without loss) is as follows, and by definition its average over all directions is 1:

$$D(\theta, \phi) = \frac{4\pi U(\theta, \phi)}{\oint U d\Omega}, \quad \frac{1}{4\pi} \oint D d\Omega = 1$$

With half-power beamwidth θ_{HP} in degrees, the factor K of the estimate $D \approx K/\theta_{\text{HP}}^2$ is found from the actual D as $K = D\theta_{\text{HP}}^2$. The check integrates rotationally symmetric patterns over the sphere, and for the circular aperture pattern $(2J_1(u)/u)^2$ with $u = ka \sin \theta$, finds J_1 from its integral representation.

Question: what does gain increase? What decides the factor that turns half-power beamwidth into gain?

2. The Average over All Directions, and Shapes Without Skirts

Computation 2.1: eight patterns.

pattern	directivity D	average over all directions	half-power beamwidth (deg)	K (sq deg)
short dipole	1.500000	1.000000000	(broad in all directions)	-
half-wave dipole	1.640922	1.000000000	(broad in all directions)	-
$\cos^2 \theta$	6.000000	1.000000000	90.0000	48600.0
$\cos^{10} \theta$	22.000000	1.000000000	42.1747	39131.5
$\cos^{100} \theta$	202.000000	1.000000000	13.4765	36686.7
$\cos^{1000} \theta$	2001.999999	1.000000000	4.2661	36435.5
circular aperture $ka = 10$	100.672828	1.000000000	18.6035	34841.9
circular aperture $ka = 30$	901.400125	1.000000000	6.1770	34392.7

The average over all directions differed from 1 by at most 4.6×10^{-15} , and the dipoles differed from 1.5 and 1.6409 by 2.8×10^{-14} and 3.8×10^{-7} .

Computation 2.2: 4π steradians is 41252.961 square degrees, and the Gaussian limit $16 \ln 2$ square radians is **36407.5** square degrees (ratio 0.882542).

Observation 2.3

gain adds no power; it measures how the 4π of solid angle is shared out. For every pattern the average over all directions is 1, and whatever exceeds 1 in some directions falls below 1 in others: for $\cos^n \theta$, which has no skirts, the factor K came down to the Gaussian limit as the beam narrowed.

3. Shapes With Skirts

Computation 3.1: the factor for a uniform circular aperture is 34841.9 at $ka = 10$ and **34392.7** square degrees at $ka = 30$, both below the Gaussian limit of 36407.5 square degrees. The fraction of power outside the first dark ring ($u = 3.8317$) is $J_0^2 + J_1^2 = \mathbf{0.162215}$ (J_1 there was 3.0×10^{-12}).

Observation 3.2

a shape with power in its skirts has less gain at the same half-power beamwidth. The half-power beamwidth sees only how narrow the main beam is: power that leaves the main beam reduces the main beam's share of the average of 1 over all directions.

4. The Separator

Observation 4.1

the separator is the fraction of power outside the main beam. Where that fraction is 0, as for $\cos^n \theta$, the factor came down to the Gaussian limit; for the circular aperture it went below: the ratio of actual gain to the estimate of 41253 over the squared beamwidth moved from 1.178097 for $\cos^2 \theta$ through 0.889310 for $\cos^{100} \theta$ to 0.833703 for the circular aperture at $ka = 30$.

Observation 4.2

what is not said. Antenna design, loss, beams of different widths in two planes, and which rule of thumb should be used are not treated. What is shown is that gain is a ratio of how 4π is shared out, and that the factor turning beamwidth into gain changes with the power in the skirts.

5. Roll Call and Closing

1. $D = 4\pi U / \oint U d\Omega$ (Section 1): average 1 over all directions.
2. average differs by 4.6×10^{-15} (Computation 2.1): eight patterns.
3. half-wave dipole **1.640922** (Computation 2.1): the known value.
4. K from 48600.0 to **36435.5** (Computation 2.1): changes with the pattern.
5. Gaussian limit **36407.5** sq deg (Computation 2.2): shapes without skirts.
6. circular aperture **34392.7** sq deg, **0.162215** outside (Computation 3.1): a shape with skirts.
7. fraction of power outside the main beam (Observation 4.1): the separator; the scope note of Paper 384; Paper 335.

Closing: Antenna gain averages to 1 over all directions, and the factor that turns beamwidth into gain depends on the power in the skirts. For all eight patterns the average of gain over all directions was 1, and gain measured how the 4π of solid angle is shared out without adding power: the factor that turns half-power beamwidth into gain came down to the Gaussian limit of 36407.5 square degrees for shapes without skirts, and was smaller still, 34392.7 square degrees, for a circular aperture that sends 0.162215 outside its first dark ring. At the same half-power beamwidth, power that leaves the main beam reduces the main beam's share: the separator is the fraction of power outside the main beam. Placed among the earlier papers: the directivity that Paper 384 did not enter was counted as the content of the gain in the round-trip equation of Paper 335. To be honest, which factor should be used is not said. That gain is a ratio of sharing out, and that the factor moves with the skirts, and no further.

References

1. Balanis, C. A. (2016). *Antenna Theory: Analysis and Design*. Wiley.
2. Kraus, J. D. (1988). *Antennas*. McGraw-Hill.
3. Yamagishi, Y. (2026). *The Round Trip Changes the Exponent* [Paper 335]. YOI: <https://www.yamagishiyuuki.com/yoi/335>; DOI: 10.5281/zenodo.22681851
4. Yamagishi, Y. (2026). *"Free-Space Loss" Is Not a Loss: The Vacuum Absorbs Nothing* [Paper 384]. YOI: <https://www.yamagishiyuuki.com/yoi/384>; DOI: 10.5281/zenodo.22699073

— *From thinking to syncing.* —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)