

In Peg Solitaire an Invariant Sum Narrows Where the Last Peg Can Remain to Five Holes, and It Can Actually Be Left in All Five

— two sums that jumps leave unchanged only rule out places the last peg cannot remain, and whether it can be left there is unknown until a sequence is found and replayed — the separator is whether a place is ruled out by an invariant or confirmed by a sequence of moves — [Paper 925]

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Abstract

English peg solitaire starts with only the centre of 33 holes empty; pegs jump over neighbours to remove them, aiming to leave a single peg. This paper narrows where the last peg can remain using two sums that jumps leave unchanged, searches for sequences that actually leave one peg in those places, and confirms them by replaying. No new theorem or law is claimed. First, there are two sums that jumps leave unchanged: with ω in the field of four elements, the two sums over peg positions (x, y) of ω to the power $x + y$ and of ω to the power $x - y$ are unchanged by all 76 kinds of jump; in 1998 trials on random positions, no difference was found (Section 2). Second, the last peg can remain in only five places: single-peg positions with the same sums as the start (centre empty) are the centre and the middle holes at the tips of the four arms, 5 places. Single-peg positions fall into 9 classes of sums (Section 2). Third, states with matching sums are a sixteenth of all, but that is not the reachable fraction: the map listing the sums has rank 4, so of the 2^{33} states exactly $1/16$ have the starting sums. The sums only rule out unreachable states and do not say that the rest are reachable (Section 3). Fourth, a single peg can actually be left in all five places: a depth-first search remembering dead ends found sequences leaving the last peg at the centre and at one arm tip (10813 dead ends remembered); replaying these sequences mapped by the board's symmetries, one peg remained after 31 moves in all five places (Section 4). Fifth, the separator is whether a place is ruled out by an invariant or confirmed by a sequence of moves: the last peg never remains where the sums do not match, but whether it can be left where they match is not told by the sums. Starting from the centre of this board, all five places allowed by the sums were reached (Section 5). **Keywords:** peg solitaire, invariant, finite field, necessary condition, search, board symmetry.

Scope of this paper (scope note): No new theorem or law is claimed: the invariants of peg solitaire by sums in the field of four elements (the “rule of three” of Conway and others) and the fact that on the English board a game started at the centre can end with one peg at the centre are both known. Shortest move counts and numbers of solutions are not treated: one sequence is found for each place; how many solutions exist and how many moves they take when consecutive jumps are grouped are not treated. Only the English 33-hole board is treated: other boards such as the French 37-hole board, and starts with a hole other than the centre empty, are not treated. Not finding is not called nonexistence: no search was cut off, and sequences were found for all five places. That the last peg does not remain where the sums fail to match is stated from the sums, not from search. Relation to earlier papers: Paper 601 showed that half of the positions of the fifteen puzzle can never be reached. Here too a quantity unchanged by moves rules out unreachable positions, and whether the places left over are actually reached is confirmed by replaying sequences. Paper 602 showed that who wins at Nim is decided by addition without carries. Here the same addition without carries is used in the field of four elements, which has two binary digits. What is added: testing 1998 times that the two sums are unchanged by jumps, narrowing where the last peg can remain to five places, deriving from the rank that matching states are 1/16, and separating that from the reachable fraction, confirming by replay that one peg can be left in all five places, and placing the separator on “ruled out by an invariant, or confirmed by a sequence”.

1. Notation and the Question

Setting: place the centre of the board at $(0, 0)$; the holes are the 33 points with $-3 \leq x, y \leq 3$ and $|x| \leq 1$ or $|y| \leq 1$. In the field of four elements $\{0, 1, \omega, \omega^2\}$ ($1 + \omega + \omega^2 = 0$, with addition without carries), assign to a set of pegs S the two sums:

$$A(S) = \sum_{(x,y) \in S} \omega^{x+y}, \quad B(S) = \sum_{(x,y) \in S} \omega^{x-y}$$

A jump removes two pegs and places one among three holes in a line, so the change in a sum is $\omega^s(1 + \omega + \omega^2) = 0$. The check compares the sums before and after jumps, lists single-peg places with the starting sums, searches for sequences by depth-first search remembering dead-end positions, and replays each sequence found from the start.

Question: where can the last peg remain, and can it actually be left in those places?

2. Unchanged Sums

Computation 2.1: there are 76 kinds of jump, and in 1998 trials on random positions the sums differed before and after a jump 0 times. The start with the centre empty has sums $(A, B) = (1, 1)$.

Computation 2.2: single-peg positions fall into 9 classes of sums, with 3, 3, 3, 3, 4, 4, 4, 4 and 5 holes. Those with the starting sums $(1, 1)$ are $(0, 0)$, $(3, 0)$, $(-3, 0)$, $(0, 3)$ and $(0, -3)$, 5 places. As an example of a mismatch, $(-3, -1)$

has sums (ω^2, ω) .

Observation 2.3

the last peg can remain in only five places, narrowed by the unchanged sums. In the twenty-eight places where the sums do not match, no sequence of jumps leaves a single peg. This follows not from search but from the sums being unchanged by jumps.

3. A Sixteenth Is Not the Reachable Fraction

Computation 3.1: listing the sums of each of the 33 holes as four binary digits gives a map of rank 4, so exactly 1/16 of the 2^{33} states have sums (1, 1).

Observation 3.2

states with matching sums are a sixteenth, but that is not the fraction of reachable states. The sums give a necessary condition that rules out unreachable states; whether a matching state is reachable from the start is not decided by the sums: reachability can only be confirmed by finding a sequence.

4. Finding and Replaying Sequences

Computation 4.1: searches were run only for (0, 0) and (3, 0), and both found sequences in 0.1 s (10813 dead ends remembered, 0 cut-offs). The starting position is unchanged by the 8 symmetries of the board, so the sequences found were mapped by symmetry.

place of the last peg	source of the sequence	result of replay
(0, 0)	search	one peg left after 31 moves
(3, 0)	search	one peg left after 31 moves
(−3, 0)	the (3, 0) sequence mapped by symmetry	one peg left after 31 moves
(0, 3)	the (3, 0) sequence mapped by symmetry	one peg left after 31 moves
(0, −3)	the (3, 0) sequence mapped by symmetry	one peg left after 31 moves

Observation 4.2

all five places allowed by the sums were actually reached. Each sequence was replayed move by move from the start, confirming that every jump was allowed and that a single peg remained at the given place: starting from the centre of this board, the places narrowed by the necessary condition and the places actually reached coincided.

5. The Separator

Observation 5.1

the separator is whether a place is ruled out by an invariant or confirmed by a sequence of moves. That the last peg does not remain where the sums fail to match follows from the sums being unchanged by jumps alone: that it can be left where they match does not follow from the sums, and can be said only after finding and replaying a sequence. In the fifteen puzzle of Paper 601, too, a quantity unchanged by moves ruled out the unreachable half.

Observation 5.2

what is not said. Shortest move counts, numbers of solutions, other boards and starts other than the centre are not treated. What is shown is that unchanged sums narrow the last peg's place to five, and that sequences reaching all five exist.

6. Roll Call and Closing

1. $A(S)$, $B(S)$ and $1 + \omega + \omega^2 = 0$ (Section 1): **unchanged sums.**
2. no difference in 1998 trials (Computation 2.1): **unchanged by jumps.**
3. **5** places where the last peg can remain (Computation 2.2): **matching sums.**
4. rank 4, matching states $1/16$ (Computation 3.1): **not the reachable fraction.**
5. one peg after 31 moves in all five (Computation 4.1): **confirmed by replay.**
6. ruled out, or confirmed (Observation 5.1): **the separator; Papers 601 and 602.**

Closing: In peg solitaire an invariant sum narrows where the last peg can remain to five holes, and it can actually be left in all five. The two sums in the field of four elements are unchanged by jumps, and on the board started at the centre the matching single-peg places are the centre and the middle holes at the tips of the four arms: states with matching sums are $1/16$ of all, but that is what remains after ruling out unreachable states, not the reachable fraction. By search and the board's symmetry, sequences were found for all five places, and replaying them left one peg after 31 moves: the separator is whether a place is ruled out by an invariant or confirmed by a sequence of moves. Placed among the earlier papers: the pattern of Paper 601, where an unchanged quantity rules out an unreachable half, was counted in peg solitaire, and the addition without carries of Paper 602 was extended to the field of four elements. To be honest: neither the shortest solution nor the number of solutions is claimed. On this board, the places ruled in and the places reached coincided, and no further.

References

1. Beasley, J. D. (1985). *The Ins and Outs of Peg Solitaire*. Oxford University Press.
 2. Berlekamp, E. R., Conway, J. H., & Guy, R. K. (1982). *Winning Ways for Your Mathematical Plays*. Academic Press.
 3. Yamagishi, Y. (2026). *Half of the Fifteen Puzzle's Positions Can Never Be Reached* [Paper 601]. YOI: <https://www.yamagishiyuuki.com/yoi/601>; DOI: 10.5281/zenodo.22805752
 4. Yamagishi, Y. (2026). *Who Wins at Nim Is Decided by Addition Without Carries* [Paper 602]. YOI: <https://www.yamagishiyuuki.com/yoi/602>; DOI: 10.5281/zenodo.22805951
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