

Bending Stiffness Sharpens Overtones Past Whole Multiples, and at the Same Pitch a Thicker String Strays Further

— an ideal string restored by tension alone has overtones at whole multiples, but with bending stiffness the higher-numbered tones are pushed sharper, and at the same pitch the coefficient grows as the square of the diameter —the separator is whether the restoring force of the string is counted from tension alone or with bending stiffness too — [Paper 942]

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Abstract

An ideal string is restored by tension alone, so its overtone frequencies are exact whole multiples of the fundamental. This paper uses the overtone formula for a steel string with bending stiffness, counts the departure from whole multiples in cents, compares two strings of different thickness tuned to the same pitch, and checks the formula against eigenvalues of a finite-difference model. No new theorem or law is claimed. First, with bending stiffness, the higher-numbered tones are pushed further above whole multiples: a steel string 0.648 m long and 1.0 mm in diameter tuned to 330 Hz has coefficient $B = 2.046219 \times 10^{-4}$, and the tones $n = 2, 5, 10$ and 20 were 0.5311, 4.2397, 17.3565 and **67.9222** cents above whole multiples (Section 2). Second, tuned to the same pitch, a thinner string strays less: at 0.5 mm diameter $B = 5.115549 \times 10^{-5}$ and the tone $n = 20$ strayed **17.4894** cents. The ratio of coefficients was **4.000000**, proportional to the square of the diameter (Section 2). Third, a thicker string at the same pitch also carries more tension: the tension was 1127.710 N at 1.0 mm and 281.927 N at 0.5 mm. As thickness grows, bending stiffness grows faster than tension (Section 2). Fourth, the formula matched finite-difference eigenvalues: solving the eigenvalues of the string supported at both ends with 1500 difference points, the frequencies for $n = 1, 2, 5, 10$ and 20 matched the formula to a relative difference below one ten-thousandth, 6864.2502 Hz against 6864.7893 Hz at $n = 20$ (Section 3). Fifth, the separator is whether the restoring force of the string is counted from tension alone or with bending stiffness too: with tension alone the wave speed does not depend on frequency and the overtones line up at whole multiples. Bending stiffness restores short waves more strongly, so higher-numbered tones go sharper (Section 4). **Keywords:** inharmonicity, bending stiffness, string vibration, overtones, cents, Young's modulus.

Scope of this paper (scope note): No new theorem or law is claimed: the overtone formula for a string with bending stiffness, the fact that its coefficient is proportional to the fourth power of the diameter and inversely to the tension and the square of the length, and inharmonicity, overtones above whole multiples as in the piano, are all known. These are set values: the length 0.648 m and steel density 7850 kg/m³ are the same as in Paper 857, while the pitch 330 Hz and Young's modulus 200 GPa are set here. The tensions correspond to no instrument's string. Instruments and tuning are not treated: wound strings, real pianos and guitars, tuning that widens intervals to match the overtone departures, and how the tone is heard are not treated. End conditions are restricted: both ends are supported free to rotate, and the motion of bridge and soundboard is not included. Relation to earlier papers: Paper 857, in its scope note, did not include bending stiffness and did not treat how a string's bending rigidity moves overtones from the ideal; this paper sits in that seat (no earlier occupant). Paper 466, in its scope note, also did not discuss instruments and said that the practice of tuning, inharmonicity and adaptive temperaments were not its subject; this paper sits in the inharmonicity part of that seat and does not enter the practice of tuning. Paper 299 showed that a string's overtones are in whole-number ratios while a circular membrane's are irrational. Here it is counted how far those whole-number ratios of a string break down with bending stiffness. What is added: the coefficient and cents for two thicknesses, a numerical check that at the same pitch the coefficient is proportional to the square of the diameter, a check of the formula against finite-difference eigenvalues, and placing the separator on "counting the restoring force from tension alone, or with bending stiffness too".

1. Notation and the Question

Setting: a round string of length L , diameter d , density ρ and Young's modulus E is stretched with tension T . Its linear density is $\mu = \rho\pi d^2/4$ and its second moment of area $I = \pi d^4/64$. Supported at both ends free to rotate, the frequency of tone n is:

$$f_n = nf_0\sqrt{1 + Bn^2}, \quad f_0 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}, \quad B = \frac{\pi^3 Ed^4}{64TL^2}$$

Tuned to the same f_0 , $T = \mu(2Lf_0)^2$, so

$$B = \frac{\pi^2 Ed^2}{64\rho L^4 f_0^2}$$

and the coefficient is proportional to the square of the diameter. The departure from whole multiples is counted in cents against n times the actual first tone f_1 : $c_n = 1200 \log_2(f_n/(nf_1))$. The set values are $L = 0.648$ m, $\rho = 7850$ kg/m³ (as in Paper 857), $E = 200$ GPa and $f_0 = 330$ Hz.

Question: how far does bending stiffness move overtones from whole multiples, and how does that change with the string's thickness?

2. Departures for Two Thicknesses

Computation 2.1: two strings tuned to 330 Hz.

diameter	linear density kg/m	tension N	B	$n = 2$	$n = 5$	$n = 10$	$n = 20$ (cents)
1.0 mm	6.165376×10^{-3}	1127.710	2.046219×10^{-4}	0.5311	4.2397	17.3565	67.9222
0.5 mm	1.541344×10^{-3}	281.927	5.115549×10^{-5}	0.1328	1.0620	4.3725	17.4894

The ratio of coefficients (1.0 mm / 0.5 mm) was **4.000000**.

Observation 2.2

the higher the number, the sharper the tone, and at the same pitch the thicker string strays further. For the 1.0 mm string, 0.5311 cents at $n = 2$ grew to 67.9222 cents at $n = 20$: for the 0.5 mm string tuned to the same 330 Hz it was 17.4894 cents at $n = 20$, and the coefficients differed by 4.000000, as the square of the diameter. The thicker string also carries more tension, 1127.710 N, a ratio of 4.000000 to the thinner one, but the ratio of bending stiffness EI was 16.000000, outpacing the growth of tension.

3. Checking by Finite-Difference Eigenvalues

Computation 3.1: the 1.0 mm string was discretized with 1500 points and the eigenvalues of $\mu \ddot{y} = Ty'' - EIy''''$ were solved.

n	difference eigenvalue Hz	formula Hz	relative difference
1	330.0337	330.0338	1.83×10^{-7}
2	660.2696	660.2700	7.31×10^{-7}
5	1654.2074	1654.2149	4.59×10^{-6}
10	3333.5296	3333.5917	1.86×10^{-5}
20	6864.2502	6864.7893	7.85×10^{-5}

Observation 3.2

the formula matched eigenvalues found by solving the string's equation of motion directly. For five tone numbers the relative difference was below 10^{-4} : the relative difference grew with the tone number. No comparison with a different spacing was made.

4. The Separator

Observation 4.1

the separator is whether the restoring force of the string is counted from tension alone or with bending stiffness too. In a string restored by tension alone, the wave speed does not depend on frequency and the overtones line up at whole multiples: bending stiffness adds a restoring force that depends on how sharply the string bends, so, as the term Bn^2 in the formula shows, it acts more on higher-numbered tones. At $n = 20$ the departure was 67.9222 cents at 1.0 mm and 17.4894 cents at 0.5 mm.

Observation 4.2

what is not said. Wound strings, real instruments, tuning that widens intervals to match the departures, the motion of bridge and soundboard, and how the tone is heard are not treated. What is shown is that for a steel string with set values, bending stiffness moves the overtones sharp of whole multiples, and at the same pitch the coefficient grows as the square of the diameter.

5. Roll Call and Closing

1. overtone formula and coefficient B (Section 1): the square of the diameter at the same pitch.
2. **67.9222** cents at $n = 20$ for 1.0 mm (Computation 2.1): higher tones stray more.
3. **17.4894** cents at 0.5 mm, coefficient ratio **4.000000** (Computation 2.1): thinner strays less.
4. finite-difference eigenvalues and the formula (Computation 3.1): relative difference below 10^{-4} .
5. tension alone, or bending stiffness too (Observation 4.1): the separator; the scope notes of Papers 857 and 466; Paper 299.

Closing: Bending stiffness sharpens overtones past whole multiples, and at the same pitch a thicker string strays further. A steel string 0.648 m long tuned to 330 Hz had its $n = 20$ tone 67.9222 cents above the whole multiple at 1.0 mm diameter and 17.4894 cents above at 0.5 mm: the ratio of coefficients was 4.000000, proportional to the square of the diameter. The formula matched finite-difference eigenvalues to a relative difference below one ten-thousandth. The separator is whether the restoring force of the string is counted from tension alone or with bending stiffness too. Placed among the earlier papers. It was counted how far the whole-number ratios of a string shown in Paper 299 break down under the bending stiffness that Papers 857 and 466 set aside. To be honest: the sound of real instruments and how to tune them are not claimed. The size of the departure and its change with thickness were counted for a string with set values, and no further.

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