

The Cheapest Number of Tanks in Series Scales with the Decades to Remove, and Each Tank's Share Is Set by the Cost Exponent Alone

— if the cost of one tank rises with its volume to a power below 1, the progress of reaction in one tank is fixed regardless of the required remainder —the separator is whether the cost of one tank is proportional to its volume or rises more gently — [Paper 960]

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— *From thinking to syncing.* —

Abstract

The more a well-mixed vessel is split into tanks in series, the smaller the total size needed to react down to the same remainder (Paper 931). This paper computes, for each number of tanks, the total size needed to bring a first-order reaction down to a remainder of 0.1, 0.01 and 0.001, and, with the cost of one tank rising as its volume to the power α (a set value), searches for the number of tanks with the lowest total cost. No new theorem or law is claimed. First, splitting into more tanks brings the needed size toward plug flow: to reach a remainder of 0.001, the product of rate constant and total residence time needed was 999 with 1 tank, 61.245553 with 2 and 9.952623 with 10, approaching plug flow's 6.907755 (Section 2). Second, if cost is proportional to volume, there is no cheapest number of tanks: with $\alpha = 1$ the cost is the total size itself, which kept falling as tanks were added from 1 to 2000, and at 2000 tanks was still 0.011943 above plug flow (remainder 0.001) (Section 2). Third, if cost rises more gently, a cheapest number exists and scales with the decades of remainder: for remainders 0.1, 0.01 and 0.001, the cheapest number was 2, 4 and 6 tanks for $\alpha = 0.6$ and 5, 10 and 15 for $\alpha = 0.8$, and the lowest costs were exactly 1 : 2 : 3. At remainder 0.001 the cost was 0.151133 and 0.038913 of one tank's (Section 3). Fourth, each tank's share is set by the cost exponent alone: the optimal product x of rate constant and residence time in one tank does not depend on the remainder; it solves $\alpha(1+x)\ln(1+x) = x$, giving 8.314868, 2.084104, 0.590761, 0.239311 and 0.109022 for $\alpha = 0.4, 0.6, 0.8, 0.9$ and 0.95, with 1.031804, 2.044450, 4.960194, 10.731874 and 22.251750 tanks per decade (Section 4). Fifth, the separator is whether the cost of one tank is proportional to its volume or rises more gently: if proportional, tanks get cheaper the more are added, with plug flow as the lower limit. If more gentle, each tank's share is fixed, and the number of tanks is set by the number of decades required (Section 5). **Keywords:** continuous stirred-tank reactor, tanks in series, first-order reaction, plug flow, equipment cost, economies of scale.

Scope of this paper (scope note): No new theorem or law is claimed: the remainder of a first-order reaction in continuous stirred tanks in series, that more tanks approach plug flow, that equipment cost rises with size to a power below 1 (economies of scale), and the design question of choosing the number of tanks by cost, are all known in chemical engineering. The cost exponent is set: the cost of one tank is set to its volume to the power a , trying $a = 0.4, 0.6, 0.8, 0.9, 0.95$ and 1. Real equipment costs, running costs, mixing power, piping and heat transfer are not treated. Only first-order reaction is treated: as in Paper 931, a first-order reaction with fixed flow rate and rate constant. Second-order reactions, tanks at different temperatures, and designs with tanks of unequal size are not included. Integer proportionality is seen at three remainders: proportionality to the number of decades is stated from the continuous formula, and checked for integer numbers of tanks at the three remainders 0.1, 0.01 and 0.001. Relation to earlier papers: Paper 931, in its scope note, did not discuss design, leaving out which number of tanks is best and cost; this paper sits in that seat (no earlier occupant). Paper 931 showed that at the same mean residence time, splitting a well-mixed vessel into more tanks in series lets less leave early and brings reaction close to plug flow. Here, with the remainder fixed, the trade-off between needed size and number of tanks is counted in cost, and how the cheapest number of tanks is set is derived. What is added: setting out the total size needed at three remainders, showing that no cheapest number exists when cost is proportional to volume, showing that when cost rises more gently the cheapest number scales with the decades of remainder, deriving the condition that fixes each tank's share by the cost exponent alone, and placing the separator on "proportional to volume, or more gentle".

1. Notation and the Question

Setting: N well-mixed tanks of equal size in series carry a first-order reaction (rate constant k). With residence time τ_1 per tank and $x = k\tau_1$, the exit remainder is $r = (1 + x)^{-N}$, and the total size needed (the product of rate constant and total residence time) is

$$k\tau = Nx = N \left(r^{-1/N} - 1 \right)$$

(for $N = 1$ to 500 the recovered remainder differed by at most a relative 5.190×10^{-14}). As N grows, $k\tau$ approaches plug flow's $\ln(1/r)$. With the flow rate fixed, the volume of one tank is proportional to x , so the cost of one tank is taken as x^a and the total cost as

$$C = Nx^a = \ln \frac{1}{r} \cdot \frac{x^a}{\ln(1+x)}$$

(using $N = \ln(1/r)/\ln(1+x)$).

Question: with the remainder fixed, how is the number of tanks with the lowest cost set?

2. Needed Size, and Cost Proportional to Volume

Computation 2.1: total size needed, $k\tau$.

remainder	$N = 1$	2	3	5	10	plug flow
0.1	9.000000	4.324555	3.463304	2.924466	2.589254	2.302585
0.01	99.000000	18.000000	10.924767	7.559432	5.848932	4.605170
0.001	999.000000	61.245553	27.000000	14.905359	9.952623	6.907755

Computation 2.2: for $a = 1$ the cost is $k\tau$ itself: 2.356427, 4.823910 and 7.407681 at $N = 50$, and 2.303911, 4.610476 and 6.919698 at $N = 2000$. For all three remainders it kept falling as tanks were added from $N = 1$ to 2000, and at $N = 2000$ was still 0.001326, 0.005306 and 0.011943 above plug flow.

Observation 2.3

if cost is proportional to volume, more tanks are always cheaper. For $a = 1$, $C/\ln(1/r) = x/\ln(1+x)$, which approaches 1 as x approaches 0. There is no cheapest number of tanks, and the lower limit of cost is plug flow's $\ln(1/r)$.

3. More Gentle Cost, and the Cheapest Number of Tanks

Computation 3.1: cheapest number found searching 1 to 500 tanks (all below 50, not at the edge of the range). In brackets, the cost and its ratio to the cost of one tank.

cost exponent	remainder 0.1	0.01	0.001
$a = 0.4$	1 (2.408225, 1.000000)	2 (4.816449, 0.766431)	3 (7.224674, 0.456029)
$a = 0.6$	2 (3.176705, 0.850024)	4 (6.353410, 0.403298)	6 (9.530115, 0.151133)
$a = 0.8$	5 (3.255602, 0.561355)	10 (6.511203, 0.164874)	15 (9.766805, 0.038913)

The cost of one tank was 2.408225, 6.284259 and 15.842590 for $a = 0.4$, 3.737193, 15.753647 and 63.057869 for $a = 0.6$, and 5.799546, 39.491912 and 250.987672 for $a = 0.8$. For all three values of a , the cheapest number and cost at remainders 0.01 and 0.001 were exactly 2 and 3 times those at 0.1 (within 10^{-9}).

Observation 3.2

a more gentle cost has a cheapest number of tanks, scaling with the decades of remainder. The cheapest number grew as the remainder was made smaller and as a approached 1 (nine cases): and the x of one tank at the cheapest number was, regardless of remainder, 9.000000 for $a = 0.4$, 2.162278 for $a = 0.6$ and 0.584893 for $a = 0.8$.

4. Each Tank's Share

Computation 4.1: the part of C other than $\ln(1/r)$, $x^a / \ln(1+x)$, contains no remainder, so the optimal x does not depend on the remainder. Setting the derivative to 0 gives $a(1+x) \ln(1+x) = x$; its solution, and the minimum found directly on a grid of 400001 values of x equally spaced in logarithm from 10^{-4} to 10^4 :

a	x (grid value)	remainder per tank	tanks per decade	cost per decade
0.4	8.314868 (8.314957)	0.107355	1.031804	2.407350
0.6	2.084104 (2.084107)	0.324243	2.044450	3.176349
0.8	0.590761 (0.590772)	0.628630	4.960194	3.255579
0.9	0.239311 (0.239310)	0.806900	10.731874	2.963072
0.95	0.109022 (0.109023)	0.901695	22.251750	2.710215

For all five values of a the solution and the grid minimum agreed within a relative 10^{-4} .

Observation 4.2

each tank's share is set by the cost exponent alone, and is smaller the closer a is to 1. For $a = 0.6$ it is cheapest for one tank to bring the remainder down to 0.324243, and for a remainder of 0.001, taken as continuous, 6.133351 such stages are stacked (the integer cheapest is 6 tanks): as a goes from 0.4 toward 0.95, the x of one tank falls from 8.314868 to 0.109022, and the tanks per decade rise from 1.031804 to 22.251750.

5. The Separator

Observation 5.1

the separator is whether the cost of one tank is proportional to its volume or rises more gently. If proportional ($a = 1$), cost approaches plug flow as x is made smaller, and there is no cheapest number of tanks (Computation 2.2): if more gentle ($a < 1$), $x^a / \ln(1 + x)$ grows like x to the power $a - 1$ as x approaches 0, and one optimal x is fixed. The number of tanks is $\ln(1/r) / \ln(1 + x)$, proportional to the number of decades of remainder required.

Observation 5.2

what is not said. Real exponents of equipment cost, running cost, mixing power, piping, second-order reactions and designs with tanks of unequal size are not treated. What is shown is how, for equal tanks and a first-order reaction with cost set as a power of volume, the cheapest number of tanks and each tank's share are set.

6. Roll Call and Closing

1. needed size and the cost formula (Section 1): cost is the log of the remainder times a part in x alone.
2. from 999 to 9.952623 at remainder 0.001 (Computation 2.1): approaching plug flow's 6.907755.
3. $a = 1$ still falling at 2000 tanks (Computation 2.2): no cheapest number.
4. 5, 10, 15 tanks at $a = 0.8$, cost ratio **0.038913** (Computation 3.1): scales with decades.
5. x of one tank **2.084104** at $a = 0.6$ (Computation 4.1): independent of remainder.
6. proportional to volume, or more gentle (Observation 5.1): the separator; the scope note of Paper 931.

Closing: The cheapest number of tanks in series scales with the decades to remove, and each tank's share is set by the cost exponent alone. With the cost of one tank rising as its volume to the power a , total cost split into the log of the remainder times a part depending only on the x of one tank: for $a = 0.6$ and 0.8 the cheapest number of tanks for remainders 0.1, 0.01 and 0.001 was 2, 4, 6 and 5, 10, 15, with optimal x per tank of 2.084104 and 0.590761. For $a = 1$ the cost kept falling up to 2000 tanks. The separator is whether the cost of one tank is proportional to its volume or rises more gently. Placed among the earlier papers: to the effect of the number of tanks that Paper 931 counted at equal residence time, the cost that Paper 931 did not treat was added, and the cheapest number of tanks

derived. To be honest: the cost exponent is a set value, and real plants also carry running costs and piping. For equal tanks and a first-order reaction, how the cheapest number is set, and no further.

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