

When the Three Errors in Tree Volume Are Correlated, the Volume Error Shifts, and the Form Factor and Height Pair Matters Most

— the variance moved by a correlation is proportional to the product of two contributions, and with the three correlations chosen the first-order error vanishes —the separator is whether the three contributions (relative error times the power in the formula) can form a triangle — [Paper 963]

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— *From thinking to syncing.* —

Abstract

The volume of a standing tree measured without felling stacks three errors, in diameter, height and form factor, and with the three independent the relative error is 0.102470 (Paper 891). This paper puts correlations into the three errors and counts how far the relative error of volume shifts and which pair's correlation matters most, in three ways: the first-order formula, the exact value with normally distributed errors, and random trees. No new theorem or law is claimed. First, the pair that matters most is form factor and height: the variance moved per unit of correlation was 0.0080 for form factor and height, 0.0064 for diameter and form factor, and 0.0040 for diameter and height, against an independent variance of 0.0105 (Section 2). Second, a correlation of ± 0.5 moves the relative error from 0.786796 to 1.175139 times independent: with a correlation of +0.5 between form factor and height the relative error was 0.120416 (1.175139 times independent), with -0.5 it was **0.080623** (0.786796 times), and for diameter and height 0.111803 and 0.092195 (Section 2). Third, a correlation in one pair alone does not double the error: the largest reached by one pair was **1.327368** times, at +1 for form factor and height. Halving it needed a form-factor and height correlation of -0.984375 , and all three at +1 gave 0.170000 (1.659030 times) (Section 3). Fourth, with the three correlations chosen, the first-order error is zero: the contributions 0.04, 0.05 and 0.08 can form a triangle ($0.04 + 0.05 = 0.09$ exceeds 0.08), and with correlations of 0.575000 for diameter and height, -0.859375 for diameter and form factor and -0.912500 for form factor and height, the first-order formula was 0. The exact value with normally distributed errors was 0.006486 (**0.063181** times independent), not zero (Section 4). Fifth, the separator is whether the three contributions can form a triangle: diameter enters squared, so its contribution is 0.04 and a triangle can be formed. If diameter were counted to the first power, the contributions 0.02, 0.05 and 0.08 could not form a triangle, and however the correlations were chosen the first-order formula did not fall below 0.010000 (Section 5). **Keywords:** tree volume, error propagation, correlation, form factor, relative error, Gauss-Hermite quadrature.

Scope of this paper (scope note): No new theorem or law is claimed: that the relative error of a product gains covariance terms when correlations are included (the error propagation formula), that positive correlation increases the error and negative correlation reduces it, and that a correlation matrix must be positive semidefinite, are all known in the treatment of measurement uncertainty. Correlation values are set: the correlation values are set here, and no claim is made about which forest or measuring method produces them. Only random errors are treated: the three errors are normal with mean 0; bias in measurement (systematic error) and species differences in form factor are not included. Only one tree is treated: one tree with diameter 0.30 m, height 20.0 m and form factor 0.45, as in Paper 891; sampling a whole stand is not treated. Relation to earlier papers: Paper 891, in its scope note, did not include correlation, taking the three errors as independent; this paper sits in that seat (no earlier occupant). Paper 891 showed that with the three errors independent the relative error of volume is 0.102470, with variance shares of 0.152381 for diameter, 0.238095 for height and 0.609524 for form factor. Here it is counted, when correlations are put into those three, which pair shifts the error by how much and how far it can vanish. Paper 938 and Paper 957 counted sampling error when a forest is divided into strata: the errors here are errors in three quantities of one tree, not errors from how samples are taken. What is added: setting out the variance moved per unit of correlation for three pairs, finding the first-order and exact values for nine correlations, finding the largest reached by one pair and the correlation that halves the error, showing that three correlations can make the first-order error vanish, and the role of the power on diameter, and placing the separator on “whether the three contributions can form a triangle”.

1. Notation and the Question

Setting: as in Paper 891, volume is $V = f \cdot \frac{\pi D^2}{4} \cdot h$, with diameter $D = 0.30$ m, height $h = 20.0$ m and form factor $f = 0.45$ carrying errors with relative standard deviations $\varepsilon_D = 0.02$, $\varepsilon_h = 0.05$ and $\varepsilon_f = 0.08$. Diameter enters squared, so the contributions to the relative error of volume are $a_D = 2\varepsilon_D = 0.04$, $a_h = 0.05$ and $a_f = 0.08$. With correlations ρ_{Dh} , ρ_{fh} and ρ_{Df} , the first-order formula is

$$\left(\frac{\sigma_V}{V}\right)^2 \approx a_D^2 + a_h^2 + a_f^2 + 2\rho_{Dh}a_Da_h + 2\rho_{fh}a_fa_h + 2\rho_{Df}a_Da_f$$

The three errors are taken as correlated normal variables, and the mean and mean square of $V/V_0 = (1 + \varepsilon_D z_D)^2(1 + \varepsilon_h z_h)(1 + \varepsilon_f z_f)$ are found by Gauss-Hermite quadrature (6 points in each direction) to give the exact standard deviation (the integrand is a polynomial, so 8 points gave the same, 0.102653410 and 0.102653410, within 10^{-12}). The random check made 400000 trees.

Question: when the three errors are correlated, how far does the relative error of volume shift, and how low can it go?

2. How Correlation Acts

Computation 2.1: independent: first-order 0.102470, exact 0.102653. Variance shares: diameter 0.152381, height 0.238095, form factor 0.609524. The variance moved per unit of correlation, $2a_i a_j$, is 0.0080 for form factor and height, 0.0064 for diameter and form factor, and 0.0040 for diameter and height (independent variance 0.0105).

Computation 2.2: relative error for nine correlations.

correlation	first-order	ratio to independent	exact
form factor and height +0.5	0.120416	1.175139	0.120655
form factor and height −0.5	0.080623	0.786796	0.080782
diameter and form factor +0.5	0.117047	1.142262	0.117326
diameter and form factor −0.5	0.085440	0.833809	0.085560
diameter and height +0.5	0.111803	1.091089	0.112113
diameter and height −0.5	0.092195	0.899735	0.092250
form factor and height +1	0.136015	1.327368	0.136330
form factor and height −1	0.050000	0.487950	0.050217
all three +1	0.170000	1.659030	0.171177

In all nine the exact value and the first-order formula differed by 0.002 or less.

Computation 2.3: with 400000 random trees the relative error was 0.102625 when independent (exact 0.102653, difference −0.000029), 0.080812 at −0.5 for form factor and height (0.080782, difference 0.000030), and 0.170836 with all three at +1 (0.171177, difference −0.000342), all three within 0.002 of the exact value.

Observation 2.4

the pair that matters most is form factor and height, whose contributions have the largest product. The order of effect was form factor and height, diameter and form factor, diameter and height, the same as the order of $2a_i a_j$: at correlations of ± 0.5 , the relative error moved to 1.175139 and 0.786796 times independent for form factor and height, and to 1.091089 and 0.899735 times for diameter and height.

3. What One Pair Can Do

Computation 3.1: the largest reached by a correlation in one pair alone is 1.327368 times, at +1 for form factor and height. Halving the relative error with the form-factor and height correlation alone needs a correlation of -0.984375 , giving a first-order value of 0.051235. With all three at +1 it was 0.170000 (1.659030 times).

Observation 3.2

a correlation in one pair alone does not reach double, and halving needs a strongly negative correlation. Even with the most effective pair at +1, the error is 1.327368 times, short of double: halving needed the form-factor and height correlation to go as negative as -0.984375 .

4. A Triangle of Three Contributions

Computation 4.1: the first-order formula is the squared length of the sum of vectors of lengths a_D , a_h and a_f at angles whose cosines are the correlations. Since $0.04 + 0.05 = 0.09$ exceeds 0.08, the three can form a triangle, and the correlations that make the sum zero are

$$\rho_{Dh} = \frac{a_f^2 - a_D^2 - a_h^2}{2a_D a_h}, \quad \rho_{Df} = \frac{a_h^2 - a_D^2 - a_f^2}{2a_D a_f}, \quad \rho_{fh} = \frac{a_D^2 - a_h^2 - a_f^2}{2a_h a_f}$$

namely 0.575000, -0.859375 and -0.912500 . The smallest eigenvalue of the correlation matrix was -1.24×10^{-16} (positive semidefinite), the first-order formula was 0, and the exact value with normal errors was 0.006486 (0.063181 times the independent exact value).

Computation 4.2: searching two in-plane angles on a 1441×1441 grid, the lowest first-order value reachable by choosing correlations was 0.000091 for contributions 0.04, 0.05 and 0.08, and 0.010000 ($0.08 - 0.05 - 0.02 = 0.01$) for the contributions 0.02, 0.05 and 0.08 obtained if diameter were counted to the first power.

Observation 4.3

choosing three correlations makes the first-order error vanish, but not the exact value. When the three contributions can form a triangle, the first-order formula goes down to 0: even then the exact value stays at 0.006486, 0.063181 times independent. With diameter counted to the first power, no triangle can be formed and the lowest was 0.010000.

5. The Separator

Observation 5.1

the separator is whether the three contributions (relative error times the power in the formula) can form a triangle. If they can, three correlations can be chosen to make the first-order error vanish: if they cannot, it does not go all the way down: with diameter counted to the first power it did not fall below 0.010000, which is $0.08 - 0.05 - 0.02 = 0.01$. The diameter error of 0.02 is the smallest of the three, but entering squared makes its contribution 0.04, and that allowed the triangle.

Observation 5.2

what is not said. Which correlations occur in real forests, bias in measurement, species differences in form factor, and sampling error of whole stands are not treated. What is shown is how far, with set correlations, the relative error of volume shifts, and how low the first-order and exact values can go.

6. Roll Call and Closing

1. first-order formula with correlation, and the exact value (Section 1): contributions 0.04, 0.05, 0.08.
2. 0.0080, 0.0064, 0.0040 per unit of correlation (Computation 2.1): form factor and height matter most.
3. **0.080623** at -0.5 for form factor and height (Computation 2.2): within 0.002 of exact and random.
4. largest for one pair **1.327368** times (Computation 3.1): short of double.
5. exact value **0.063181** times independent at the triangle correlations (Computation 4.1): first-order is 0.
6. whether the three contributions can form a triangle (Observation 5.1): the separator; the scope note of Paper 891.

Closing: When the three errors in tree volume are correlated, the volume error shifts, and the form factor and height pair matters most. The variance moved per unit of correlation was 0.0080 for form factor and height, 0.0064 for diameter and form factor and 0.0040 for diameter and height, and a form-factor and height correlation of ± 0.5 gave relative errors of 0.120416 and 0.080623: one pair alone reached at most 1.327368 times, but the contributions 0.04, 0.05 and 0.08 can form a triangle, and with the three correlations chosen the first-order formula was 0 and the exact value fell to 0.063181 times independent. The separator is whether the three contributions can form a triangle. Placed among the earlier papers: into the three errors that Paper 891 counted as independent, the correlations that Paper

891 did not include were put. To be honest: the correlation values are set, and which correlations real measurements produce was not examined. With set correlations, how far the error shifts and how far it can vanish, and no further.

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