On a String with Stray Overtones, Which Overtones Are Matched Decides How Wide the Octave Is

— on a plain string, matching the 2nd overtone widens the octave by 4.3481 cents and matching the 6th by 37.8711 cents, and unmatched pairs beat —the separator is whether the octave is counted by the fundamentals or by the numbers of the matched overtones — [Paper 974]

Yuuki YAMAGISHI (山岸 勇輝)
Yamagishi Complexity Torus Lab, Japan
September 2026
YOI: https://www.yamagishiyuuki.com/yoi/974

— *From thinking to syncing.* —

Abstract

On a string with bending stiffness, overtones stray sharp of whole multiples, and on a wound string the departure is small (Papers 942 and 958). Tuning that widens intervals to match those departures was not treated there.Using the stiffness coefficients of the wound and plain strings of Paper 958, this paper tunes an upper note to a lower note of 110 Hz by matching the lower 2nd overtone to the upper fundamental, the lower 4th to the upper 2nd, or the lower 6th to the upper 3rd, and counts how far the octave widens and how fast the other pairs beat.No new theorem or law is claimed. First, whichever overtones are matched, the octave widens: **on the plain string (stiffness coefficient 1.681419×10^{-3}), matching the 2nd, 4th and 6th overtones made the ratio of fundamentals 4.3481, 17.1773 and 37.8711 cents wider than 2** (Section 2). **Second, unmatched pairs beat**: **with the plain string matched at the 2nd overtone, the lower 4th and upper 2nd beat 3.291949 times per second and the lower 6th and upper 3rd 13.034495 times. Matched at the 6th, the lower 2nd and upper fundamental beat 4.315949 times** (Section 2). **Third, on the wound string the same is smaller by orders**: **on the wound string (5.163611×10^{-5}) the widenings were 0.1341, 0.5361 and 1.2054 cents, and the 4th-overtone beat when matched at the 2nd was 0.102204 per second** (Section 2). **Fourth, the widening is set by the stiffness coefficient and the overtone numbers alone**: **the widening matched the closed form of 600 times the base-2 logarithm of $(1 + B\,m^2)/(1 + B\,k^2)$ cents (lower mth, upper kth) within 10^{-9} cents, and the widening when matched at the 2nd overtone matched the 2nd-overtone departures of Paper 958, 0.1341 and 4.3481 cents** (Section 3). **Fifth, the separator is whether the octave is counted by the fundamentals or by the numbers of the matched overtones**: **counted by fundamentals, the octave is one ratio of 2. Counted by matched overtones, each choice of number gives a different widening, and matching at the 6th widened it 8.709748 times as much as matching at the 2nd (plain string)** (Section 4). **Keywords**: tuning, octave stretch, inharmonicity, overtones, beats, string vibration.

Scope of this paper (scope note): **No new theorem or law is claimed**: the overtone formula for a string with bending stiffness, tuning octaves wide to match overtone departures (stretch tuning), and that beats are the difference of two frequencies, are all known in the physics of instruments and in tuning.**Both strings have the same stiffness coefficient**: the lower and upper strings are given the same coefficient, using 5.163611×10^{-5} for the wound string and 1.681419×10^{-3} for the plain string of Paper 958. That the coefficient changes from note to note on real pianos and guitars is not included.**The merit of a tuning is not stated**, which overtones should be matched, how beats sound, and timbre are not treated.**Only one octave is treated**: only the pair of a 110 Hz lower note and its upper note; accumulating the widening across a keyboard is not treated.**Relation to earlier papers**: **Paper 958, in its scope note, did not treat instruments and tuning, leaving out real guitars and pianos, tuning that widens intervals to match the departures, and how the tone is heard; this paper sits in the seat of tuning that widens intervals** (no earlier occupant).**Paper 942, in its scope note, also left out tuning that widens intervals to match overtone departures; this paper sits in that seat as well**.**Paper 958 showed that a wound string carries its weight in the winding and its overtones stray far less than a plain string of the same weight**. **Here it is counted how far those departures keep the octave from being settled by one choice of matching.Paper 466 showed that everything cannot be tuned at once is a proved fact**. **Here it is counted that with stray overtones, not all overtone pairs can be matched even within one octave.**What is added: **setting out widening and beats for three matchings**, **comparing two strings**, **checking with a closed form**, **confirming agreement with the departures of Paper 958**, and **placing the separator on "fundamentals, or overtone numbers"**.

1. Notation and the Question

Setting: **as in Paper 942, the nth overtone of a string with stiffness coefficient B is $f_n = n f_0 \sqrt{1 + B n^2}$ (f_0 the fundamental without stiffness). For a lower note with $f_0 = 110$ Hz, the upper note's f_0' is set so that its kth overtone equals the lower mth ($m = 2k$). The octave widening is the departure of the actual ratio of fundamentals from 2:**

$$\Delta = 1200 \log_2 \frac{f_1'}{2 f_1} = 600 \log_2 \frac{1 + B m^2}{1 + B k^2} \text{ cents}$$

The beat of an unmatched pair (m', k') is taken as the difference $|f_{m'} - f_{k'}'|$ of its two frequencies.

Question: **on a string with stray overtones, can the octave be tuned in one way?**

2. Three Matchings

Computation 2.1: **plain string (diameter 0.955522 mm, $B = 1.681419 \times 10^{-3}$).**

matching	upper fundamental (Hz)	widening (cents)	beat 2:1	beat 4:2	beat 6:3
lower 2 = upper 1	220.738585	4.3481	0.000000	**3.291949**	13.034495
lower 4 = upper 2	222.380430	17.1773	1.641846	0.000000	8.075997
lower 6 = upper 3	225.054534	**37.8711**	4.315949	5.361657	0.000000

Computation 2.2: wound string ($B = 5.163611 \times 10^{-5}$).

matching	upper fundamental (Hz)	widening (cents)	beat 2:1	beat 4:2	beat 6:3
lower 2 = upper 1	220.022719	0.1341	0.000000	0.102204	0.408684
lower 4 = upper 2	220.073817	0.5361	0.051098	0.000000	0.255358
lower 6 = upper 3	220.158918	1.2054	0.136200	0.170217	0.000000

Beats are in Hz (times per second). In all six cases the matched pair's beat was below 10^{-9} Hz.

> **Observation 2.3**
>
> with one matching, the other pairs do not match.Matching the plain string at the 2nd overtone left 4:2 and 6:3 beating, and matching at the 6th left 2:1 and 4:2 beating: the higher the matched number, the larger the widening, and on the wound string the same pattern was smaller by orders.

3. The Closed Form

Computation 3.1: the closed form $600 \log_2((1 + Bm^2)/(1 + Bk^2))$ gave 4.3481, 17.1773 and 37.8711 cents for 2:1, 4:2 and 6:3 on the plain string and 0.1341, 0.5361 and 1.2054 cents on the wound string, all six within 10^{-9} cents of the values obtained without it. The widening for 2:1 matched the 2nd-overtone departures of Paper 958 (wound 0.1341, plain 4.3481 cents) within 5×10^{-5}. The 6:3 widening was 8.709748 times (plain) and 8.990718 times (wound) the 2:1 widening.

Observation 3.2

the widening is set by the stiffness coefficient and the overtone numbers alone.Regardless of the lower note's pitch, B with m and k fixes the widening: the widening when matched at the 2nd overtone was exactly the departure of the 2nd overtone from a whole multiple.

4. The Separator

Observation 4.1

the separator is whether the octave is counted by the fundamentals or by the numbers of the matched overtones.Counted by the ratio of fundamentals, the octave is a single ratio of 2: counted by matching overtones, each number gives a different widening; on the plain string it opened from 4.3481 cents at the 2nd to 37.8711 cents at the 6th, 8.709748 times as much. Matching any one pair left the others beating.

Observation 4.2

what is not said.Note-by-note stiffness coefficients of real pianos or guitars, accumulation across a keyboard, which matching is better, and how beats or timbre sound are not treated. What is shown is that for two strings with the same stiffness coefficient, the octave widening is not settled in one way but depends on the numbers of the matched overtones.

5. Roll Call and Closing

1. **overtone and widening formulas** (Section 1): **set by B and numbers alone.**
2. **plain-string widening 4.3481, 17.1773, 37.8711 cents** (Computation 2.1): **differs by matching.**
3. **matched at the 2nd, 4:2 beats 3.291949 Hz** (Computation 2.1): **other pairs do not match.**
4. **wound string 0.1341, 0.5361, 1.2054 cents** (Computation 2.2): **smaller by orders.**
5. **agrees with the closed form and Paper 958** (Computation 3.1): **check.**
6. **fundamentals, or overtone numbers** (Observation 4.1): **the separator; the scope notes of Papers 958 and 942.**

Closing: On a string with stray overtones, which overtones are matched decides how wide the octave is.Tuning an octave above 110 Hz on the plain string of Paper 958, matching at the 2nd, 4th and 6th overtones widened it by

4.3481, 17.1773 and 37.8711 cents, and matching at the 2nd left the 4:2 pair beating 3.291949 times per second: on the wound string the same widenings were 0.1341, 0.5361 and 1.2054 cents, set by a closed form in the stiffness coefficient and overtone numbers alone.The separator is whether the octave is counted by the fundamentals or by the numbers of the matched overtones.Placed among the earlier papers: with the overtone departures counted in Papers 958 and 942, the interval-widening tuning they did not treat was counted, showing that the "cannot tune everything" of Paper 466 occurs even within one octave.To be honest. This is a single pair with the same coefficient on both strings, and real instrument tuning was not examined. How much the octave differs with the matched overtone numbers, and no further.

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Yuuki YAMAGISHI (山岸 勇輝)

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